Homotopy Analysis for Periodic Motion of Time-delayed Duffing system

You Xiangcheng
School of Petroleum Engineering, China University of Petroleum
Beijing 102249, China
E-mail: applexcyou@163.com

Abstract—In this paper, the periodic motions of local dynamics of time-delayed oscillators near a single Hopf bifurcation have been investigated by means of the homotopy analysis method (HAM). With this technique, analytical approximations with high accuracy for all possible solutions are captured, which match the numerical solutions in the whole time regions. Two examples of dynamic systems are considered, which focus on the periodic motions near a Hopf bifurcation of an equilibrium point. It is found that the current technique lead to higher accurate prediction on the local dynamics of time-delayed systems near a Hopf bifurcation than the energy analysis method or the traditional method of multiple scales with strongly nonlinear examples. We studied the temporal dynamics of time-delayed systems in various regimes characterized by the parameters of the oscillator and the time delay parameter. The results given in this paper show that the time delay plays very important role in the analysis of multiply periodic motions with time-delayed systems. This paper is presented a general approach to the analysis of periodic motions of time-delayed systems. Although here we only consider a non-autonomous Duffing system with linear and nonlinear time-delayed position feedback, HAM can be extended to solve other time-delayed systems, such as coupled oscillators with time-delayed, feedback control which may have significance for the control of some physical or engineering systems.

Keywords-Homotopy analysis method; Periodic motion; Time-delayed; Duffing system; Delayed differential equation

I. INTRODUCTION

In many applications, the time delays involved in nonlinear dynamics systems have to be considered even if they are very short. It is shown that the study of dynamic behavior usually get wrong conclusions because of simply ignoring small delays, moreover, some mechanical phenomena can be explained reasonably only considering the existence of time delay. The evolution of a time-delayed system depends on both the current and previous state of the system. So it is reasonable to describe the time-delayed dynamic systems by delayed differential equations (DDEs). On the other hand, the dynamics of time-delayed systems has also obtained great attention from the researchers in other fields such as machine tool dynamics, neural networks and biology, medicine and population dynamics [1-3].

Many studies [4,5] on the time-delayed systems have been done over the past several decades. Among these researches, the Van der Pol-Duffing oscillator has drawn considerable attention since it serves as a simple model in various engineering fields. For instance, some Van der Pol-Duffing oscillators with delayed feedback show the extremely simple dynamics if the time delay disappears, while infinite number of periodic motions even for very small time delays [6,7]. Furthermore, it is proved that the delayed feedback control is one of the most effective and flexible strategies in the fields of controlling chaos of nonlinear dynamics systems [8]. Obviously, the time-delay systems may exist abundant dynamics which involves chaotic motion and Hopf bifurcation [9-12]. However, the research on periodic motions is of special interest in engineering applications. Perturbation approaches, such as the method of multiple scales, the method of harmonic balance, are widely used to reveal the complex dynamics of nonlinear systems [13-16].

H. Khan et al. [17] investigated a nonlinear model in biology by means of HAM. A new discontinuous function is defined so as to express the piecewise continuous solutions of time-delay differential equations. It is shown that the proposed HAM method seems to be applicable to general systems that can be described using a general delay differential equation (DDE) of the form $x' = f\left(x, x\left(t-\tau\right)\right)$. The objective of this paper is to develop an effective analytical technique based on the homotopy analysis method (HAM, Refs.[18-23]) to give analytical approximations for periodic motion of Duffing system with delayed feedback. A Duffing oscillator with time-delayed feedback described by the second-order DDEs is used as an example to propose a general analytic approach for nonlinear time-delayed dynamic systems.

II. ANALYTICAL APPROXIMATIONS

Adding the terms of time-delayed position feedback in a Duffing system

$$\ddot{x} + \hat{\alpha}\dot{x} + \hat{\omega}_0^2 x + \hat{\beta}x^3 = \hat{A}x\left(t-\tau\right) + \hat{B}x^3\left(t-\tau\right). \quad (1)$$

Where $\hat{\alpha}$ is damping coefficient, $\hat{\omega}_0$ is system natural frequency, $\hat{\beta}$ is rigidity coefficient, $\hat{A}$ is the feedback-gain coefficient and τ is time-delay. $\hat{A}, \hat{B} > 0$ denotes positive feedback and $\hat{A}, \hat{B} < 0$ denotes negative feedback.

The initial conditions of Eq.(1) are

$$x(t)=a_0, \dot{x}(t)=0, -\tau \le t < 0 \quad (2)$$

The system to be considered is the time-delayed position feedback control system, here we assume that the system has no signal feedback when t<0.

Under the transformations

$$\theta=\omega t, x(t)=au(\theta) \quad (3)$$

Where ω is frequency. Eq.(1) becomes

$$\begin{aligned}&\omega^2 \ddot{u}(\theta)+\omega\hat{\alpha}\dot{u}(\theta)+\hat{\omega}_0^2 u(\theta)+\hat{\beta}a^2u^3(\theta)\\&=\hat{A}u(\theta-\omega\tau)+\hat{B}u^3(\theta-\omega\tau)\end{aligned} \quad (4)$$

Subject to the initial conditions

$$u(0)=1, \dot{u}(0)=0. \quad (5)$$

Where the prime represents differentiation with respect to θ and $T=2\pi/\omega$ is period of system.

From the physical point of view, periodic motions of time-delayed dynamics systems can be expressed by periodic functions. Obviously, $u(\theta)$ may be expressed in this form:

$$u(\theta)=\tilde{a}_0+\sum_{n=1}^{+\infty}[\tilde{a}_n\cos(n\theta)+\tilde{b}_n\sin(n\theta)] \quad (6)$$

Where $\tilde{a}_n$ and $\tilde{b}_n$ are coefficients. This provides us with the rule of solution expression for $u(\theta)$.

We choose the initial guess of $u(\theta)$ based on the initial conditions (5) as

$$u_0(\theta)=\cos\theta \quad (7)$$

Besides, we choose

$$L[f]=\frac{\partial^2 f}{\partial\theta^2}+f \quad (8)$$

As the auxiliary linear operator, which has the following property

$$L[C_1\sin\theta+C_2\cos\theta]=0 \quad (9)$$

Where C_1 and C_2 are integral constants and f is a real function. The nonlinear operators are defined based on Eq.(4) as

$$\begin{aligned}N[U(\theta;q),\Omega(q),A(q)]&=\Omega^2(q)\frac{\partial^2 U(\theta;q)}{\partial\theta^2}\\&+\hat{\alpha}\Omega(q)\frac{\partial U(\theta;q)}{\partial\theta}+\hat{\omega}_0^2U(\theta;q)\\&+\hat{B}A^2(q)U^3(\theta;q)\\&-\hat{A}U[\theta-\Omega(q)\tau;q]\\&-\hat{B}U^3[\theta-\Omega(q)\tau;q]\end{aligned} \quad (10)$$

Where $q\in[0,1]$ is the embedding parameter, $U(\theta;q)$ is a real function of θ and q, $\Omega(q)$ and $A(q)$ are the real function of q respectively.

Then, let η denote an auxiliary parameter. We construct the HAM deformation equation

$$\begin{aligned}&[1-B_1(q;c_1)]L[U(\theta;q)-u_0(\theta)]\\&=c_0A_1(q;c_2)H(\theta)N[U(\theta;q),\Omega(q),A(q)]\end{aligned} \quad (11)$$

Subject to the conditions

$$U(0;q)=1, \left.\frac{\partial U(\theta;q)}{\partial\theta}\right|_{\theta=0}=0 \quad (12)$$

Where $\theta\ge 0$ and $H(\theta)=1$. Obviously, when $q=0$ and $q=1$, it is clear from Eq.(4) and the above zero-order deformation equation that

$$\begin{aligned}&U(\theta;0)=u_0(\theta), U(\theta;1)=u(\theta)\\&\Omega(0)=\omega_0, \Omega(1)=\omega\\&A(0)=a_0, A(1)=a\end{aligned} \quad (13)$$

So, as q increases from 0 to 1, $U(\theta;q)$ varies from the initial guess $u_0(\theta)$ to the exact solution $u(\theta)$, so do $\Omega(q)$ and $A(q)$ from the initial guesses ω_0 and a_0 to the corresponding exact values ω and a. Expanding $U(\theta;q)$, $\Omega(q)$ and $A(q)$ in Taylor's series with respect to *q*. Differentiating the HAM deformation equation (11) *m* times with respect to *q*, then setting *q=0*, and finally dividing them by *m!*, the mth-order deformation equations can be used, then the analytical approximations for u, ω, a can be obtained.

III. RESULTS ANALYSIS

Many researchers have been made great efforts to investigate the stability of time-delayed dynamics systems over the past decades and many encouraging results have been obtained. However, most of these investigations were given by perturbation methods, which can hardly give results with high accuracy owing to its inherent limitation. Here, we re-examine the system of Duffing oscillator with time-delayed feedback by means of the homotopy analysis method. It is found the proposed technique can improve the accuracy for all captured solutions which are obtained from the analysis of periodic motions near a Hopf bifurcation of an equilibrium point.

When $\hat{\omega}_0 = 0, \beta = 0, \hat{A} = -1$, as an illustrative example is following

$$\ddot{x} + \hat{\alpha}\dot{x} = \hat{B}x^3(t-\tau) - x(t-\tau). \tag{14}$$

By means of the homotopy analysis method, the accurate analytical approximation of $x(t)$ is obtained, and the 10th-order approximation given by $\hat{\alpha} = -0.9, \hat{B} = -2,$ $\tau = 2$ and $\eta = -0.01$ reads

$$\begin{aligned} x(t) = {} & 0.57039\cos(\omega t) + 0.00114\cos(3\omega t) \\ & - 3.96212\times10^{-7}\cos(5\omega t) \\ & + 8.42288\times10^{-11}\cos(7\omega t) \\ & + 5.07591\times10^{-16}\cos(9\omega t) \\ & + 5.69177\times10^{-18}\cos(11\omega t) \end{aligned} \tag{15}$$

Where the frequency ω equals to 0.51545 in this case. As shown in Fig. 1, the 10th HAM approximation of periodic motion agrees well with the numerical results in the large region of time *t*. With 4-order Runge-Kutta numerical method, the computational domain of *t*, ranged from 0 to 1000, is divided into 1000000 intervals, namely fixed time step is 0.001. The convergence criterion used is based on the Root Mean Square error (RMS) which is 1×10^{-6} in the present work. It is worth noting that the approximate solution by using HAM contains an auxiliary parameter η, which provides a simple way to adjust and control the convergence region and rate of series solution. Mathematically, $x(t)$ is dependent on both of the physical variable *t* and the auxiliary parameter η when other parameters are given. So, from mathematical view points, given a value of *t*, $x(t)$ is a power series of η and thus its convergence is determined by η. For example, in the case of $\hat{\alpha} = -0.9, \hat{\omega}_0 = 0, \hat{\beta} = 0,$ $\hat{A} = -1, \hat{B} = -2, \tau = 2$, regarding η as a variable, we can plot the curves of $x(t) \sim \eta$ when *t=0*, as shown in Fig. 1.

As shown in TABLE I, the 10th HAM approximations of the amplitude of the bifurcation periodic solutions are in a very good agreement with numerical solutions. It can be found that the nonlinear becomes stronger and the amplitude becomes smaller as the value of parameters increasing. Furthermore, the approximate solutions are only more reliable with small parameters. As the value of parameters increases, the approximate solutions are gradually inaccurate. And the formula of approximation prediction is applied, which is a local method and it may fail for some DDEs. For example, the amplitude *a* simply equals to $2\sqrt{(\hat{\alpha} - \sin\tau)/3\hat{B}\sin\tau}$ in this case.

IV. CONCLUSION

In this paper, the periodic motions of the local dynamics of time-delayed oscillators near a single Hopf bifurcation have been investigated by means of the homotopy analysis method. With this technique, analytical approximations with high accuracy for all possible solutions are captured, which match the numerical solutions in the whole time regions. We studied the temporal dynamics of time-delayed systems in various regimes characterized by the parameters of the oscillator and the time delay parameter. The results given in this paper show that the time delay plays very important role in the analysis of multiply periodic motions with time-delayed systems.

It is well known that the time-delayed systems exhibit complex dynamics, including periodic, quasi-periodic and chaotic motions. This paper is presented a general approach to the analysis of periodic motions of time-delayed systems. Although here we only consider a non-autonomous Duffing system with linear and nonlinear time-delayed position feedback, the homotopy analysis method can be extended to solve other time-delayed systems, such as coupled oscillators with time-delayed, feedback control which may have significance for the control of some physical or engineering systems.

A. Figures and Tables

TABLE I. AMPLITUDE OF THE BIFURCATED PERIODIC SOLUTION OF EQ.(14)

Bifurc.param.	***Approx.sol.***	***Numer.sol.***	***S10th HAM sol.***
$\tau = 0.1, \hat{\alpha} = 0.05, \hat{B} = -0.1$	2.5798	2.7044	2.7045
$\tau = 0.1, \hat{\alpha} = 0.05, \hat{B} = -0.5$	1.1537	1.2094	1.2094
$\tau = 0.1, \hat{\alpha} = 0.05, \hat{B} = -2.0$	0.5769	0.6047	0.6047
$\tau = 1.0, \hat{\alpha} = 0.5, \hat{B} = -0.1$	2.3261	2.5944	2.5945
$\tau = 1.0, \hat{\alpha} = 0.5, \hat{B} = -0.5$	1.0403	1.1602	1.1602

Bifurc.param.	***Approx.sol.***	***Numer.sol.***	***S10th HAM sol.***
$\tau = 1.0, \hat{\alpha} = 0.5, \hat{B} = -2.0$	0.5201	0.5801	0.5801
$\tau = 2.0, \hat{\alpha} = 0.9, \hat{B} = -0.1$	0.3692	2.5530	2.5530
$\tau = 2.0, \hat{\alpha} = 0.9, \hat{B} = -0.5$	0.1651	1.1417	1.1417
$\tau = 2.0, \hat{\alpha} = 0.9, \hat{B} = -2.0$	0.0826	0.5709	0.5710

a. Sample of a Table footnote. (Table footnote)

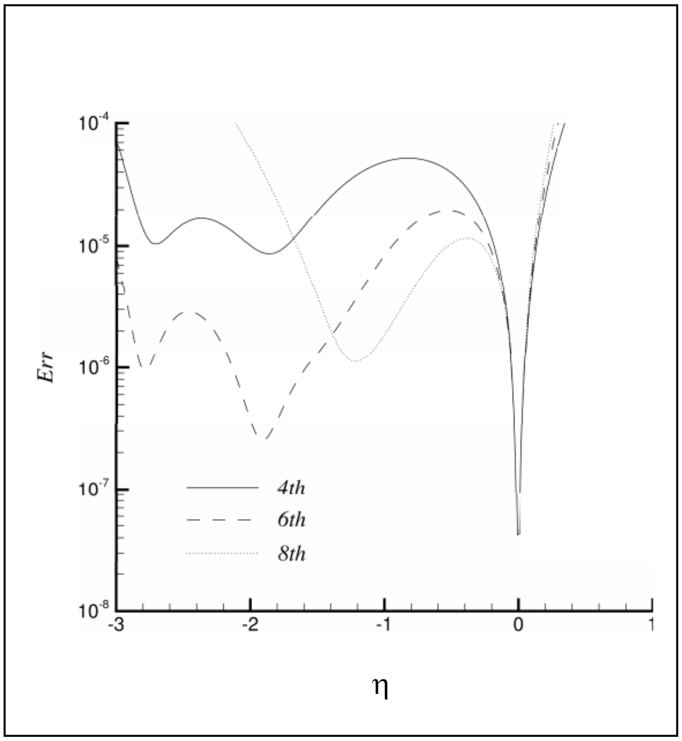

Figure 1. The curves of averaged residual error ~ η in the case of $\hat{\alpha} = 0.9, \hat{B} = -2, \tau = 2$. Solid line: 8th-order HAM solution; Dashed line: 6th-order HAM solution; Dash-dotted line: 4th-order HAM solution.

ACKNOWLEDGMENT

Thanks for the financial supports of the Science Foundation of China University of Petroleum, Beijing (No.2462015YQ0208).

REFERENCES

[1] D. Perioux, T. Erneux, A. Gavrielides, and V. Kovanis, “Hopf bifurcation subject to a large delay in a laser system,’’ SIAM J. Appl. Math., vol. 61, pp.966-982, 2000.

[2] Shayer, and S.A. Campbell, “Stability, bifurcation, and multi-stability in a system of two coupled neurones with multiple time delays,’’ SIAM J. Appl. Math., vol. 61, pp. 673-700, 2000.

[3] N. MacDonald, “Biological Delay Systems: Linear Stability Theory,” Cambridge University Press, 1st ed., New York, 1989, pp.30-40.

[4] Y.X. Qin, Y.Q. Liu, L. Wang, and Z.X. Zheng, “Stability of dynamic systems with delays,” Science Press, 1st ed., Beijing, 1989, pp.60-70.

[5] Y. Kuang, “Delay Differential Equations with Applications to Population Dynamics,” Academic Press, 1st ed., New York, 1993, pp.10-25.

[6] F.M. Atay, “Van der pol's oscillator under delayed feedback,” J. Sound Vibr., vol.218, pp.333-339, 1998.

[7] J.C.F. de Oliveira, “Oscillations in a van der Pol equation with delayed argument,” J. Math. Analy. Appl., vol. 275, pp.789-803, 2002.

[8] S. Yamamoto, T. Hino, and T. Ushio, “Delayed feedback control with a minimal-order observer for stabilization of chaotic discrete-time systems,” Inter. J. Bifur. Chaos, vol.12, pp.1047-1055, 2002.

[9] J. Belair, and S.A. Campbell, “Stability and bifurcations of equilibria in a multiple-delayed differential equation,” SIAM J. Appl. Math., vol. 54, pp.1402-1424, 1994.

[10] D.V.R. Reddy, A. Sen, and G.L. Johnston, “Dynamics of a limit cycle oscillator under time delayed linear and nonlinear feedbacks,” Phys. D: Nonlinear Phenom., vol. 44, pp.335-357, 2000.

[11] H.J. Dai, and J. Xu, “Effects of time delay on periodic motions in nonlinear system with parametric excitation,” Chinese Quart. Mech., vol.25, pp.367-374, 2004.

[12] R. Xu, M.A.J. Chaplain, and F.A. Davidson, “Periodic solutions for a delayed predator-prey model of prey dispersal in two-patch environments,” Nonlinear Analy.: Real World Appli., vol.5,pp.183-206, 2004.

[13] B.D. Hassard, N.D. Kazarinoff, and Y.H. Wan, “Counting roots of the characteristic equation for linear-delay differential systems,” J. Differ. Eqn., vol.136, pp.222-235, 1997.

[14] Z.H. Wang, and H.Y. Hu, “Delay-independent stability of retarded dynamic systems of multiple degrees of freedom,”J. Sound Vibr., vol. 226, pp.57-81, 1999.

[15] P. Yu, Y. Yuan, and J. Xu, “Study of double Hopf bifurcation and chaos for an oscillator with time delayed feedback,” Commun. Nonlinear Sci. Numer. Simul., vol.7, pp.69-91, 2002.

[16] Z.H. Wang, and H.Y. Hu, “An energy analysis of the local dynamics of a delayed oscillator near a Hopf bifurcation,” Nonlinear Dyn., vol.46, pp.149-159, 2006.

[17] H. Khan, S.J. Liao, R.N. Mohapatra, and K. Vajravelu, “An analytical solution for a nonlinear time-delay model in biology,” Commun. Nonlinear Sci. Numer. Simul., vol.14, pp.3141-3148, 2009.

[18] S.J. Liao, “Homotopy Analysis Method in Nonlinear Differential Equations,” Springer, 1st ed., Heidelberg, pp.80-88, 2012.

[19] D.L. Xu, Z.L. Liu, S.J. Liao, and M. Stiassnie, “On the steady-state fully resonant progressive waves in water of finite depth,” J. Fluid Mech., vol.710, pp.379-418, 2012.

[20] H. Xu, and I. Pop, “Mixed convection flow of a nanofluid over a stretching surface with uniform free stream in the presence of both nanoparticles and gyrotactic microorganisms,” Int. J. Heat Mass Transf., vol.75, pp.610-623, 2014.

[21] H. Xu, and I. Pop, “Fully developed mixed convection flow in a horizontal channel filled by a nanofluid containing both nanoparticles and gyrotactic microorganisms,” Eur. J. Mech. B/Fluids, vol.46, pp.37-45, 2014.

[22] Z.L. Liu, S.J. Liao, “Steady-state resonance of multiple wave interactions in deep water,” J. Fluid Mech., vol.742, pp.664-700, 2014.

[23] D.L. Xu, Z.L. Liu, and S.J. Liao, “Equilibrium states of class-I Bragg resonant wave system,” Eur. J. Mech. B/Fluids, vol.50, pp.38-51, 2015.