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EXTENDED EXPONENTIATED POWER LINDLEY DISTRIBUTION

V. Ranjbar[1], M. Alizadeh[2], G. G. Hamedani[3]

ABSTRACT

In this study, we introduce a new model called the Extended Exponentiated Power Lindley distribution which extends the Lindley distribution and has increasing, bathtub and upside down shapes for the hazard rate function. It also includes the power Lindley distribution as a special case. Several statistical properties of the distribution are explored, such as the density, hazard rate, survival, quantile functions, and moments. Estimation using the maximum likelihood method and inference on a random sample from this distribution are investigated. A simulation study is performed to compare the performance of the different parameter estimates in terms of bias and mean square error. We apply a real data set to illustrate the applicability of the new model. Empirical findings show that proposed model provides better fits than other well-known extensions of Lindley distributions.

Key words: Power Lindley distribution, Structural properties, Failure-time, Maximum likelihood estimation.

1. Introduction

The statistical analysis and modeling of lifetime data are essential in almost all applied sciences such as, biomedical science, engineering, nance, and insurance, amongst others. A number of one-parameter continuous distributions for modelling lifetime data has been introduced in statistical literature including exponential, Lindley, gamma, lognormal, and Weibull. The exponential, Lindley and Weibull distributions are more popular than the gamma and lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. The Lindley distribution is a very well-known distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, and lifetime analysis. It was introduced by Lindley (1985) to analyze failure time data, especially in applications of modeling stress-strength reliability. The motivation for introducing the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. It may also be mentioned that the Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distributions. This distribution represents a good alternative to the exponential failure time distributions that suffer from not exhibiting unimodal and bathtub shaped failure rates (Bakouch et al. (2012)). The properties and inferential procedure for the Lindley distribution were studied by

[1]Golestan University, Gorgan, Iran. E-mail: vahidranjbar@gmail.com
[2]Persian Gulf University, Bushehr, Iran. E-mail: moradalizadeh78@gmail.com
[3]Marquette University, Milwaukee, USA. E-mail: g.hamedani@mu.edu

Ghitany et al. (2008, 2013). They show via a numerical example that the Lindley distribution gives better modeling than the one based on the exponential distribution when hazard rate is unimodal or bathtub shaped. Furthermore, Mazucheli and Achcar (2011) showed that many of the mathematical properties are more exible than those of the exponential distribution and proposed the Lindley distribution as a possible alternative to exponential or Weibull distributions. The need for extended forms of the Lindley distribution arises in many applied areas. The emergence of such distributions in the statistics literature is quite recent. For some extended forms of the Lindley distribution and their applications, the interested reader is referred to Kumaraswamy Lindley (Cakmakyapan and Ozel, (2014)), beta odd log-logistic Lindley (Cordeiro et al., (2015)), generalized Lindley (Nadarajah et al., (2011)), quasi Lindley distributions (Shanker and Mishra, (2013)).

The probability density function (pdf) and cumulative distribution function (cdf) of the power Lindley distribution are given respectively by

$$f(x) = \frac{\lambda^2}{1+\lambda}\beta x^{\beta-1}e^{-\lambda x^{\beta}},$$
$$F(x) = 1-(1+\frac{\lambda\, x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\, \mathrm{x}^{\beta}}. \tag{1}$$

It can be seen that this distribution is a mixture of Exponential and gamma distributions. Having only one parameter, the Lindley distribution does not provide enough exibility for analyzing different types of lifetime data. To increase the exibility for modeling purposes it will be useful to consider further alternatives to this distribution. Our purpose here is to provide a generalization that may be useful for more complex situations. Once the proposed distribution is quite exible in terms of pdf and hazard rate function (hrf), it may provide an interesting alternative for describing income distributions and can also be applied in actuarial science, nance, bioscience, telecommunications and modeling lifetime data. Therefore, goal is to introduce a new distribution using the Lindley distribution. Alizadeh et.al (2017), introduced a new class of exponentiated distributions which called Extended Exponentiated distribution (EE-G). The cdf and pdf of this family are given by

$$F(x;\alpha,\gamma,\xi) = \int_0^{\frac{G(x;\xi)^{\alpha}}{1-G(x;\xi)^{\gamma}}} \frac{dt}{(1+t)^2}\, dt = \frac{G(x;\xi)^{\alpha}}{G(x;\xi)^{\alpha}+1-G(x;\xi)^{\gamma}} \tag{2}$$

$$f(x;\alpha,\gamma,\xi) = \frac{g(x;\xi)\, G(x;\xi)^{\alpha-1}\, [\alpha+(\gamma-\alpha)\, G(x;\xi)^{\gamma}]}{[G(x;\xi)^{\alpha}+1-G(x;\xi)^{\gamma}]^2}, \tag{3}$$

where $\alpha,\gamma > 0$ are two shape parameters and ξ is the vector of parameters for baseline cdf G. For $\alpha = \gamma$, it contains exp-G family of distributions. Taking $G(x;\xi)$ as power Lindley distribution with parameters λ,β, we introduce a new extension of

Exponentiated power Lindley distribution.

The article is outlined as follows: In Section 2, we introduce the EE-PL distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments, and moment generating function are also obtained. Moreover, mean deviation, Lorenz and Bonferroni curves, order statistics and finally a simulation study are presented in this section. In section 3, the asymptotic properties and extreme values are obtained. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 4. The characterizations of EE-PL distribution are presented in Section 5. The Applications to real data sets are considered in Section 6. Finally, Section 7 offers some concluding remarks.

2. Main properties

2.1. Probability Density and Cumulative Distribution Functions

Inserting (1) in (2), the cdf of the EE-PL with four parameters $(\alpha,\beta,\gamma,\lambda>0)$ is defined by

$$F(x;\alpha,\beta,\gamma,\lambda)=\frac{\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}}{\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\gamma}},\quad x\geq 0 \qquad (4)$$

The corresponding pdf for $x>0$ is given by

$$\begin{aligned} f(x;\alpha,\beta,\gamma,\lambda) &= \lambda^{2}\beta x^{\beta-1}(1+x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha-1} \\ &\times\frac{\left\{\alpha+(\gamma-\alpha)\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\gamma}\right\}}{(1+\lambda)\left\{\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\gamma}\right\}^{2}}, \end{aligned} \qquad (5)$$

where λ is a scale parameter α,β and γ are the shape parameters. Here, α and β govern the skewness of (5). A random variable X with the pdf (5) is denoted by $X\sim EE-PL(\alpha,\beta,\gamma,\lambda)$. It is easy to see that:

- For $\beta=1$, we obtain Extended Generalized Lindley by Ranjbar et al. (2018).
- For $\alpha=\gamma$, we obtain Exponentiated power Lindley.
- For $\alpha=\gamma$ and $\beta=1$, we obtain Generalized Lindley.
- For $\alpha=\gamma=1$, we obtain Power Lindley.
- For $\alpha=\gamma=\beta=1$, we obtain Lindley.

Some of the possible shapes of the density function (5) for the selected parameter values are illustrated in Figure 1. As seen in Figure 1, the density function can take various forms depending on the parameter values. It is evident that the EE-LP distribution is much more flexible than the power Lindley distribution, i.e. the additional shape parameter allows

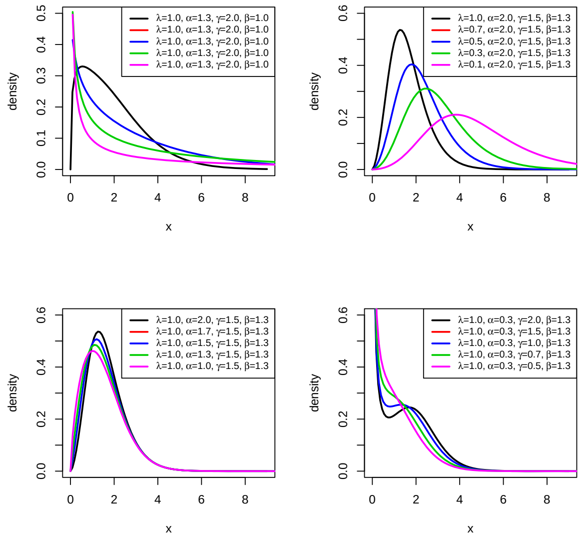

Figure 1: Plots of Pdf of the EE-PL model for selected λ, α, γ and β.

for a high degree of flexibility of the EE-PL distribution. Both unimodal and monotonically decreasing and increasing shapes appear to be possible.

2.2. Survival and Hazard Rate Functions

Central role is played in the reliability theory by the quotient of the pdf and survival function. We obtain the survival function corresponding to (4) as

$$S(x;\lambda,\alpha,\gamma,\beta) = \frac{1-\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})e^{-\lambda x^{\beta}}\right]^{\gamma}}{\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})e^{-\lambda x^{\beta}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})e^{-\lambda x^{\beta}}\right]^{\gamma}}. \tag{6}$$

In reliability studies, the hrf is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of

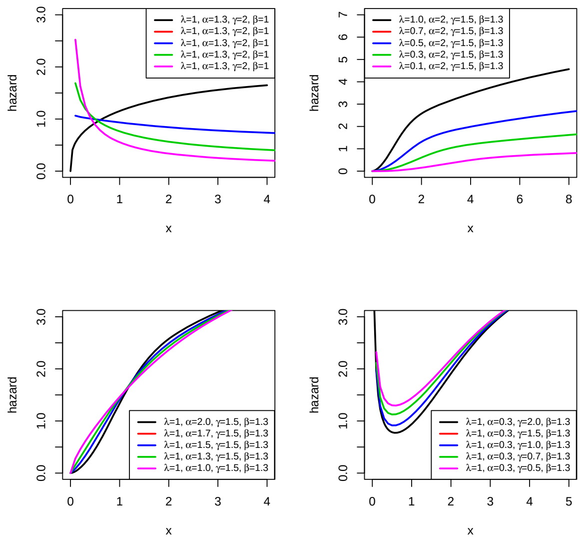

Figure 2: Hazard rate functions of the EE-PL model for selected λ, α, γ and β.

the EE-LP distribution. The hrf of X takes the form

$$\begin{aligned} h(x;\lambda,\alpha,\gamma,\beta) \quad = \quad & \lambda^2 \beta x^{\beta-1}(1+x^{\beta})\mathrm{e}^{-\lambda x^{\beta}} \left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\alpha-1} \\ & \times \left\{\alpha+(\gamma-\alpha)\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\gamma}\right\} / \\ & \left[\left\{\left(1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right)^{\alpha}+1-\left(1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right)^{\gamma}\right\}\right. \\ & \left.\times \left\{1-\left[1-(1+\frac{\lambda x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda x^{\beta}}\right]^{\gamma}\right\}\right], \quad x>0. \qquad (7) \end{aligned}$$

Plots of the hrf of the EE-PL distribution for several parameter values are displayed in Figure 2. Figure 2 shows that the hrf of the EE-PL distribution can have very flexible shapes, such as increasing, decreasing, bathtub followed by upside down bathtub, and bathtub shapes for the selected values of the model parameters. This attractive flexibility makes the hrf of the EE-PL distribution useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

2.3. Mixture representations for the pdf and cdf

In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Some useful expansions for (4) can be derived by using the concept of power series. We have

$$\begin{aligned}
[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{\alpha} &= \sum_{i=1}^{\infty}(-1)^{i}\binom{\alpha}{i}[1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{i} \\
&= \sum_{i=1}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k} \\
&= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k} \\
&= \sum_{k=0}^{\infty}a_k[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k},
\end{aligned}$$

where $a_k = a_k(\alpha) = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}$. Also

$$[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{\alpha}+1-[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{\gamma} = \sum_{k=0}^{\infty}b_k[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k},$$

where $b_0 = a_0(\alpha)+1-a_0(\gamma)$ and $b_k = a_k(\alpha)-a_k(\gamma)$ for $k \geq 1$. Then using the ratio of two power series, we can write

$$\begin{aligned}
F(x) &= \frac{\sum_{k=0}^{\infty}a_k[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k}}{\sum_{k=0}^{\infty}b_k[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k}} \\
&= \sum_{k=0}^{\infty}c_k[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k}, \qquad (8)
\end{aligned}$$

where $c_0 = \frac{a_0}{b_0}$ and for $k \geq 1$,

$$c_k = \frac{1}{b_0}[a_k - \frac{1}{b_0}\sum_{r=1}^{k}b_r c_{k-r}]. \qquad (9)$$

Equation (8) shows that we can write the cdf of EE-PL as a Linear combination of generalized lindly distribution. Then we can write

$$f(x) = \sum_{k=0}^{\infty}c_{k+1}\frac{(k+1)\lambda^2\beta x^{\beta-1}(1+x^{\beta})}{1+\lambda}e^{-\lambda x^{\beta}}[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})e^{-\lambda x^{\beta}}]^{k}.$$

2.4. Moments and Moment Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now we obtain ordinary moments and the moment generating function (mgf) of the EE-PL distribution. We define and compute

$$A(a_1,a_2,a_3,a_4;\lambda,\beta) = \int_0^{\infty}x^{a_1}(1+x^{\beta})^{a_2}\,\mathrm{e}^{-a_3 x^{\beta}}\left[1-(1+\frac{\lambda}{1+\lambda}x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a_4}\mathrm{d}x. \qquad (10)$$

Using generalized binomial expansion, one can obtain

$$A(a_1,a_2,a_3,a_4;\lambda,\beta)=\sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}(\frac{\lambda}{1+\lambda})^l\times\frac{\Gamma(\frac{a_1+1}{\beta}+k+r)}{\beta\,(\lambda\,l+a_3)^{\frac{a_1+1}{\beta}+k+r}}. \tag{11}$$

Next, the nth moment of the EE-PL distribution is given by

$$E\,[X^n]=\frac{\lambda^2\,\beta}{1+\lambda}\sum_{k=0}^{\infty}k\,c_k\,A(n+\beta-1,1,\lambda,k;\lambda,\beta). \tag{12}$$

For integer values of k, let $\mu_k'=E(X^k)$ and $\mu=\mu_1'=E(X)$, then one can also find the kth central moment of the EE-PL distribution through the following well-known equation

$$\mu_k=E(X-\mu)^k=\sum_{r=0}^{k}\binom{k}{r}\mu_r'(-\mu)^{k-r}. \tag{13}$$

The moment generating function of a random variable provides the basis of an alternative route to analytical results compared with working directly with its pdf and cdf. Using (12) and (13), we obtain

$$M_X(t)=E\left[\mathrm{e}^{\mathrm{t}X}\right]=\frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}A(k+1,\lambda,0,\lambda-t).$$

Using (13), the variance, skewness and kurtosis measures can be obtained. Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. For the EE-PL distribution, The skewness can be computed as

$$S=\frac{\mu_3}{\mu_2^{3/2}}=\frac{\mu_3'-3\mu_2'\mu_1'+2\mu_1'^3}{(\mu_2'-\mu_1'^2)^{3/2}}$$

and the kurtosis is based on octiles as

$$K=\frac{\mu_4}{\mu_2^2}=\frac{\mu_4'-4\mu_1'\mu_3'+6\mu_1'^2\mu_2'-3\mu_1'^4}{\mu_2'-\mu_1'^2}.$$

When the distribution is symmetric $S=0$, and when the distribution is right (or left) skewed $S>0(or\ S<0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

We present first four ordinary moments, skewness and kurtosis of the EE-PL distribution for various values of the parameters in Table 1. Plots for skewness and kurtosis, when $\lambda=2$, are presented in Figure 3.

Next, we define and compute

$$B(a_1,a_2,a_3,a_4;y,\lambda,\beta)=\int_0^y x^{a_1}\,(1+x^{\beta})^{a_2}\,\mathrm{e}^{-a_3\,x^{\beta}}\left[1-(1+\frac{\lambda}{1+\lambda}\,x^{\beta})\mathrm{e}^{-\lambda x^{\beta}}\right]^{a_4}\mathrm{d}x. \tag{14}$$

Table 1: Moments, skewness, and kurtosis of the EE-PL dist. for the some parameter values.

λ	α	β	γ	μ'_1	μ'_2	μ'_3	μ'_4	*Skewness*	*Kurtosis*
2.0	0.5	0.5	0.5	0.457	1.788	16.97	289.197	7.4082	164.75
2.0	0.5	0.5	1.0	0.413	0.457	0.783	1.7888	2.3382	3.052
2.0	0.5	0.5	3.0	0.608	0.468	0.413	0.4004	0.2704	0.243
2.0	0.5	1.0	0.5	0.752	3.384	33.41	575.819	5.6285	172.40
2.0	0.5	1.0	1.0	0.571	0.752	1.413	3.3844	1.7824	3.070
2.0	0.5	1.0	3.0	0.688	0.592	0.571	0.5947	0.0234	0.261
2.0	0.5	2.0	0.5	1.176	6.223	65.03	1142.245	4.3521	182.28
2.0	0.5	2.0	1.0	0.750	1.176	2.445	6.2232	1.3397	3.107
2.0	0.5	2.0	3.0	0.757	0.716	0.750	0.8367	-0.1566	0.288
2.0	2.0	0.5	0.5	0.655	2.028	17.57	291.944	7.0008	156.70
2.0	2.0	0.5	1.0	0.646	0.655	0.981	2.0282	2.1655	2.572
2.0	2.0	0.5	3.0	0.818	0.708	0.646	0.619	0.4272	0.134
2.0	2.0	1.0	0.5	0.985	3.744	34.46	580.961	5.4788	167.33
2.0	2.0	1.0	1.0	0.806	0.985	1.678	3.7447	1.7687	2.701
2.0	2.0	1.0	3.0	0.882	0.822	0.806	0.8272	0.2475	0.140
2.0	2.0	2.0	0.5	1.437	6.729	66.76	1151.513	4.3391	179.76
2.0	2.0	2.0	1.0	0.986	1.437	2.781	6.7291	1.4076	2.814
2.0	2.0	2.0	3.0	0.943	0.941	0.986	1.0779	0.0508	0.150

From the generalized binomial expansion, we have

$$\begin{aligned} & B(a_1,a_2,a_3,a_4;a,\lambda,\beta) \\ &= \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}\left(\frac{\lambda}{1+\lambda}\right)^l\times\frac{\gamma\left(\frac{a_1+1}{\beta}+k+r,\frac{y^{\frac{1}{\beta}}}{\lambda\, l+a_3}\right)}{\beta\,(\lambda\, l+a_3)^{\frac{a_1+1}{\beta}+k+r}}, \end{aligned} \tag{15}$$

where $\gamma(\lambda,z)=\int_0^z t^{\lambda-1}\,\mathrm{e}^{-t}\,dt$ denotes the incomplete gamma function. Now, the nth incomplete moment of the EE-PL distribution is found to be

$$m_n(y)=E\left[X^n\,|X<y\right]=\frac{\lambda^2\beta}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,B(n+\beta-1,1,\lambda,k,y;\lambda,\beta). \tag{16}$$

2.5. Mean Deviations, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the EE-PL distribution are presented in this section. Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, L(p) can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume y, and Bonferroni curve, B(p) is the scaled conditional mean curve, that is, ratio of group mean income of the population.

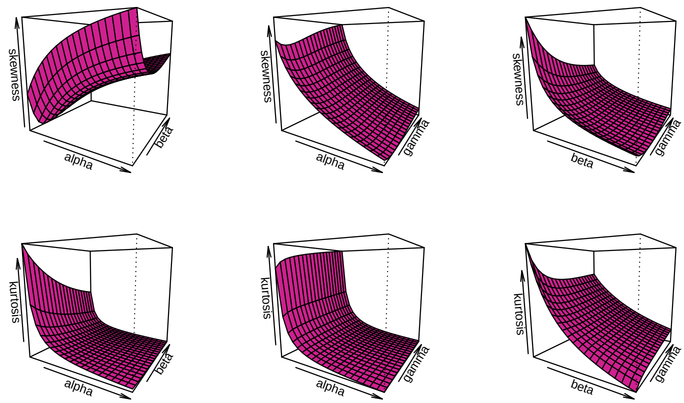

Figure 3: Values of skewness and kurtosis of EE-PL for some values of α, β and γ.

2.5.1 Mean deviations

The amount of scatter in a population may be measured to some extent by deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, defined by

$$\delta_1(X) = \int_0^\infty |x-\mu| f(x)\,\mathrm{d}x,$$

and

$$\delta_2(X) = \int_0^\infty |x-M| f(x)\,\mathrm{d}x.$$

respectively, where $\mu = E(X)$ and $M = \mathrm{Median}(X) = Q(0.5)$ denotes the median and $Q(p)$ is the quantile function. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\delta_1(X) = 2\mu F(\mu) - 2\int_0^\mu x f(x)\,\mathrm{d}x$$

and

$$\delta_2(X) = \mu - 2\int_0^M x f(x)\,\mathrm{d}x$$

Finally have

$$\delta_1(X) = 2\mu F(\mu) - \frac{\beta\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)c_{k+1}A(\beta,1,\lambda,k;\lambda,\beta),$$

and

$$\delta_2(X) = \mu - \frac{2\beta\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)c_{k+1}B(\beta,1,\lambda,k;M,\lambda,\beta).$$

2.5.2 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have applications in economics as well as other fields like reliability, medicine and insurance. Let $X \sim EE-PL(\lambda,\beta,\alpha,\gamma)$ and $F(x)$ be the cdf of X, then the Bonferroni curve of the EE-PL distribution is given by

$$B(F(x)) = \frac{1}{\mu F(x)} \int_0^x t f(t)\mathrm{d}t,$$

where $\mu = E(X)$. Therefore, from (15), we have

$$B(F(x)) = \frac{1}{\mu F(x)} \times \frac{\beta\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)c_{k+1} B(\beta,1,\lambda,k;x,\lambda,\beta).$$

The Lorenz curve of the EE-PL distribution can be obtained using the relation

$$L(F(x)) = F(x)B(F(x)) = \frac{1}{\mu} \times \frac{\beta\lambda^2}{1+\lambda} \sum_{k=0}^{\infty} (k+1)c_{k+1} B(\beta,1,\lambda,k;x,\lambda,\beta).$$

2.6. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1,\ldots,X_n$ is a random sample from any EE-PL distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = K f(x) F^{i-1}(x) \{1-F(x)\}^{n-i} = K \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x) F(x)^{j+i-1},$$

where $K = 1/B(i,n-i+1)$. We use the result of Gradshteyn and Ryzhik for a power series raised to a positive integer n (for $n \geq 1$)

$$\left(\sum_{i=0}^{\infty} a_i u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i} u^i, \tag{17}$$

where the coefficients $d_{n,i}$ (for $i = 1,2,\ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i a_0)^{-1} \sum_{m=1}^{i} [m(n+1)-i]\, a_m\, d_{n,i-m}. \tag{18}$$

We can show that the density function of the ith order statistic of any EGL distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k} f_{EPL}(x;\lambda,\beta,r+k+1), \tag{19}$$

where $f_{EPL}(x;\lambda,\beta,r+k+1)$ denotes the density function of exponentiated power Lindley distribution with parameters λ,β and $r+k+1$,

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\,c_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!\,j!}.$$

Here, c_r is given by (9) and the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = c_0^{j+i-1}$ and recursively we have:

$$f_{j+i-1,k} = (k\,c_0)^{-1} \sum_{m=1}^{k} [m\,(j+i)-k]\,c_m\,f_{j+i-1,k-m}, k \geq 1.$$

Equation (19) is the main result of this section. It reveals that the pdf of the ith order statistic is a triple linear combination of exponentiated power Lindley distributions. Therefore, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and others can be derived using this result.

2.7. Simulation study

In this section, we propose Inverse cdf method for generating random data from the EE-PL distribution. If $U \sim U(0,1)$, the solution of non-linear equation

$$u = \frac{\left[1-(1+\frac{\lambda\,x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\,x^{\beta}}\right]^{\alpha}}{\left[1-(1+\frac{\lambda\,x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\,x^{\beta}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda\,,x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\,x^{\beta}}\right]^{\gamma}} \quad (20)$$

has cdf (4).

3. Asymptotic Properties and Extreme Value

One of the main usage of the idea of an asymptotic distribution is in providing approximations to the cumulative distribution functions of the statistical estimators. Moreover, the extreme value theory is a branch of statistics dealing with the extreme deviations from the median of probability distributions. It seeks to assess, from a given ordered sample of a given random variable, the probability of events that are more extreme than any previously observed. Extreme value analysis is widely used in many disciplines,

3.1. Asymptotic properties

The asymptotic of cdf, pdf and hrf of the EE-PL distribution as $x \to 0$ are, respectively, given by

$$F(x) \sim (\lambda x^{\beta})^{\alpha} \quad as \quad x \to 0,$$
$$f(x) \sim \alpha\beta\lambda^{\alpha} x^{\alpha\beta-1} \quad as \quad x \to 0,$$
$$h(x) \sim \alpha\beta\lambda^{\alpha} x^{\alpha\beta-1} \quad as \quad x \to 0.$$

The asymptotic of cdf, pdf and hrf of the EE-PL distribution as $x \to \infty$ are, respectively, given by

$$1-F(x) \sim \frac{\gamma\lambda}{1+\lambda} x^{\beta} e^{-\lambda x^{\beta}} \quad as \quad x \to \infty,$$
$$f(x) \sim \frac{\beta\gamma\lambda^2}{1+\lambda} x^{2\beta-1} e^{-\lambda x^{\beta}} \quad as \quad x \to \infty,$$
$$h(x) \sim \beta\lambda x^{\beta-1} \quad as \quad x \to \infty.$$

These equations show the effect of parameters on the tails of the EE-PL distribution.

3.2. Extreme Value

Let $X_1,...,X_n$ be a random sample from (5) and $\bar{X} = (X_1 + ... + X_n)/n$ denote the sample mean, then by the usual central limit theorem, the distribution of $\sqrt{n}(\bar{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotic of the extreme values $M_n = \max(X_1,...,X_n)$ and $m_n = \min(X_1,...,X_n)$. For 4, it can be seen that

$$\lim_{t\to 0} \frac{F(tx)}{F(t)} = x^{\alpha\beta},$$

and

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = \mathrm{e}^{-\alpha\lambda x^{\beta}}.$$

Thus, it follows from Theorem 1.6.2 in Leadbetter et al. (1983) that there must be norming constants $a_n > 0, b_n, c_n > 0$ and d_n such that

$$Pr[a_n(M_n - b_n) \le x] \to \mathrm{e}^{-\mathrm{e}^{-\lambda\alpha x^{\beta}}},$$

and

$$Pr[a_n(m_n - b_n) \le x] \to 1 - \mathrm{e}^{-x^{\alpha\beta}},$$

as $n \to \infty$. Using Corollary 1.6.3 of Leadbetter et al. (1983), we can obtain the form of normalizing constants a_n, b_n, c_n and d_n.

4. Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider estimation of the unknown parameters of the EE-PL distribution by the method of maximum likelihood. Let $x_1, x_2, ..., x_n$ be observed values from the EE-PL distribution with parameters α, β, γ and λ. The log-likelihood function for ($\alpha;\beta;\gamma;\lambda$) is given by

$$\begin{aligned} \ell_n &= 2n\log(\lambda\beta) + (\beta-1)\sum_{i=1}^{n}\log(x_i) + \beta\sum_{i=1}^{n}\log(1+x_i) - \lambda\sum_{i=1}^{n}x_i \\ &\quad +(\alpha+1)\sum_{i=1}^{n}\log k_i + \sum_{i=1}^{n}\log(\alpha+(\gamma-\alpha)k_i^{\alpha}) - 2\sum_{i=1}^{n}\log(k_i^{\alpha}+1-k_i^{\gamma}) \end{aligned}$$

where

$$k_i = 1-(1+\frac{\lambda}{1+\lambda}x_i^{\beta})e^{-\lambda x_i^{\beta}}.$$

The derivatives of the log-likelihood function with respect to the parameters α, β, γ and λ are given respectively, by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \sum_{i=1}^{n} \log k_i + \sum_{i=1}^{n} \frac{1-k_i^{\alpha-1}(\alpha+k_i)}{\alpha+(\gamma-\alpha)k_i^{\alpha}} - 2\sum_{i=1}^{n} \frac{\alpha k_i^{\alpha-1}}{k_i^{\alpha}+1-k_i^{\alpha}} \\
\frac{\partial \ell_n}{\partial \beta} &= \frac{2n}{\beta} + \sum_{i=1}^{n} \log(x_i(x_i+1)) + (\alpha-1)\sum_{i=1}^{n} \frac{k_i^{(\beta)}}{k_i} + \sum_{i=1}^{n} \frac{\alpha(\gamma-\alpha)k_i^{\alpha-1}k_i^{(\beta)}}{\alpha+(\gamma+\alpha)k_i^{\alpha}} \\
&\quad -2\sum_{i=1}^{n} \frac{\alpha k_i^{(\beta)}k_i^{\alpha-1} - \gamma k_i^{(\beta)}k_i^{\gamma-1}}{k_i^{\alpha}+1-k_i^{\gamma}} \\
\frac{\partial \ell_n}{\partial \gamma} &= \sum_{i=1}^{n} \frac{k_i^{\alpha}}{\alpha+(\gamma-\alpha)k_i^{\alpha}} + 2\sum_{i=1}^{n} \frac{k_i^{\gamma}\log(k_i)}{k_i^{\alpha}+1-k_i^{\gamma}} \\
\frac{\partial \ell_n}{\partial \lambda} &= \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n} \frac{k_i^{(\lambda)}}{k_i} + \sum_{i=1}^{n} \frac{\alpha(\gamma-\alpha)k_i^{\alpha-1}k_i^{(\lambda)}}{\alpha+(\gamma+\alpha)k_i^{\alpha}} \\
&\quad -2\sum_{i=1}^{n} \frac{\alpha k_i^{(\lambda)}k_i^{\alpha-1} - \gamma k_i^{(\lambda)}k_i^{\gamma-1}}{k_i^{\alpha}+1-k_i^{\gamma}},
\end{aligned}$$

where

$$\begin{aligned}
k_i^{(\beta)} &= \frac{\partial k_i}{\partial \beta} = \left[1+\frac{\lambda}{\lambda+1}(x_i^{\beta}-1)\right] x_i^{\beta} e^{-\lambda x_i^{\beta}} \log x_i \\
k_i^{(\lambda)} &= \frac{\partial k_i}{\partial \lambda} = x_i^{\beta} e^{-\lambda x_i^{\beta}} \left[1+\frac{\lambda}{\lambda+1}x_i^{\beta} + \frac{1}{(1+\lambda)^2}\right].
\end{aligned}$$

The maximum likelihood estimates (MLEs) of $(\alpha;\beta;\gamma;\lambda)$, say $(\widehat{\alpha};\widehat{\beta};\widehat{\gamma};\widehat{\lambda})$, are the simultaneous solution of the equations $\frac{\partial \ell_n}{\partial \alpha}=0; \frac{\partial \ell_n}{\partial \beta}=0; \frac{\partial \ell_n}{\partial \gamma}=0; \frac{\partial \ell_n}{\partial \lambda}=0.$

For estimating the model parameters, numerical iterative techniques should be used to solve these equations.We can investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. Let $\theta = (\alpha;\beta,\gamma,\lambda)^T$, then the asymptotic distribution of $\sqrt{n}(\theta-\hat{\theta})$ is $N_4(0,K(\theta)^{-1})$, under standard regularity conditions (see Lehmann and Casella, [**?**] 1998, pp. 461-463), where $K(\theta)$ is the expected information matrix. The asymptotic behavior remains valid if $K(\theta)$) is superseded by the observed information matrix multiplied by $1/n$, say $I(\theta)/n$, approximated by $\hat{\theta}$, i.e. $I(\hat{\theta})/n$. We have

$$I(\theta) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha\gamma} & I_{\alpha\lambda} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta\gamma} & I_{\beta\lambda} \\ I_{\gamma\alpha} & I_{\gamma\beta} & I_{\gamma\gamma} & I_{\gamma\lambda} \\ I_{\lambda\alpha} & I_{\lambda\beta} & I_{\lambda\gamma} & I_{\lambda\lambda} \end{bmatrix}$$

where

$$I_{\alpha\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha^2}; \quad I_{\alpha\beta} = I_{\alpha\beta} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \beta}; \quad I_{\alpha\gamma} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \gamma}; \quad I_{\alpha\lambda} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \lambda}$$

$$I_{\beta\gamma}=I_{\gamma\beta}=\frac{\partial^2\ell_n}{\partial\beta\partial\gamma};\quad I_{\beta\lambda}=I_{\lambda\beta}=\frac{\partial^2\ell_n}{\partial\beta\partial\lambda};\quad I_{\gamma\lambda}=I_{\lambda\gamma}=\frac{\partial^2\ell_n}{\partial\gamma\partial\lambda}.$$

5. Characterizations

This section deals with various characterizations of EE-PL distribution. These characterizations are presented in four directions: (i) based on the ratio of two truncated moments; (ii) in terms of the hazard function; (iii) in terms of the reverse hazard function and (iv) based on the conditional expectation of certain function of the random variable. It should be noted that characterization (i) can be employed also when the cdf does not have a closed form. We present our characterizations $(i)-(iv)$ in four subsections.

5.1. Characterizations based on truncated moments

Our first characterization employs a theorem due to Glänzel (1986) , see Theorem 1 of Appendix A. The result, however, holds also when the interval H is not closed since the condition of Theorem 1 is on the interior of H. We like to mention that this kind of characterization based on a truncated moment is stable in the sense of weak convergence (see, Glänzel (1990)).

Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let

$$q_1(x)=\frac{\left\{1+\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}-\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\gamma}\right\}^{2}}{\left\{\alpha+(\gamma-\alpha)\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\gamma}\right\}}$$

and

$$q_2(x)=q_1(x)\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}$$

for $x>0$. The random variable X belongs to the family (5) if and only if the function η defined in Theorem1 has the form

$$\eta(x)=\frac{1}{2}\left\{1+\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right\},\quad x>0.$$

Proof. Let X be a random variable with pdf (2.2), then

$$(1-F(x))E[q_1(X)\mid X\geq x]=\frac{1}{\alpha(1+\lambda)}\left\{1-\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{\alpha}\right\}$$

and

$$(1-F(x))E[q_2(X)\mid X\geq x]=\frac{1}{2\alpha(1+\lambda)}\left\{1-\left[1-\left(1+\frac{\lambda x^{\beta}}{1+\lambda}\right)e^{-\lambda x^{\beta}}\right]^{2\alpha}\right\}.$$

Further,

$$\eta(x)q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left\{1-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha\right\}>0\ for\ \ x>0$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{\alpha\beta\lambda^2x^{\beta-1}\left(1+x^\beta\right)e^{-\lambda x^\beta}\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^{\alpha-1}}{1-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha},\quad x>0,$$

and hence

$$s(x)=-\lambda\log\left\{1-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha\right\},\quad x>0.$$

Now, according to Theorem 1, X has density (2.2).

Let $X:\Omega\to(0,\infty)$ be a continuous random variable and let q_1 be as in Proposition (5.1). Then, X has pdf (2.2) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q(x)}=\frac{\alpha\beta\lambda^2x^{\beta-1}\left(1+x^\beta\right)e^{-\lambda x^\beta}\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^{\alpha-1}}{1-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha},\quad x>0.$$

The general solution of the differential equation in Corollary (5.1) is

$$\eta(x)=\left\{1-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha\right\}^{-1}\times$$
$$\left[-\int\alpha\beta\lambda^2x^{\beta-1}\left(1+x^\beta\right)e^{-\lambda x^\beta}\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^{\alpha-1}(q_1(x))^{-1}q_2(x)\,dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition (5.1) with $D=\frac{1}{2}$.

For $\alpha=\gamma=1$, $q_1(x)\equiv1$ and $q_2(x)=e^{-\lambda x^\beta}$, we have $\eta(x)=\frac{1}{2}e^{-\lambda x^\beta}$, $x>0$, $s'(x)=\lambda\beta x^{\beta-1}$, $x>0$ and

$$\eta(x)=e^{\lambda x^\beta}\left[-\int\lambda\beta x^{\beta-1}\left(1+x^\beta\right)e^{-\lambda x^\beta}q_2(x)\,dx+D\right].$$

5.2. Characterization in terms of the hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization of EE-PL,for $\alpha = \gamma = 1$, in terms of the hazard function which is not of the above trivial form.

Let $X : \Omega \to (0, \infty)$ be a continuous random variable. Then, X has pdf (5), for $\alpha = \gamma = 1$, if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x) - (\beta - 1)x^{-1}h_F(x) = \lambda^2\beta^2x^{2(\beta-1)}\left(1+\lambda+\lambda x^{\beta}\right)^{-2},\ x > 0,$$

with the initial condition $h_F(0) = 0$ for $\beta > 1$.

Proof. If X has pdf (2.2), for $\alpha = \gamma = 1$, then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{x^{-(\beta-1)}h_F(x)\right\} = \lambda^2\beta^2\left\{\frac{1+x^{\beta}}{1+\lambda+\lambda x^{\beta}}\right\}$$

or

$$h_F(x) = \frac{\lambda^2\beta x^{\beta-1}\left(1+x^{\beta}\right)}{1+\lambda+\lambda x^{\beta}} \quad x > 0,$$

which is the hazard function of (2.2).

5.3. Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)},\ \ x \in support\ of\ F.$$

In this subsection we present characterization of EE-PL distribution in terms of the reverse hazard function.

Let $X : \Omega \to (0, \infty)$ be a continuous random variable. Then, X has pdf (2.2) if and only if its reverse hazard function $r_F(x)$ satisfies the differential equation

$$r_F'(x)+\lambda\beta x^{\beta-1}r_F(x)=$$
$$\frac{\lambda^2\beta e^{-\lambda x^\beta}}{1+\lambda}\frac{d}{dx}\left\{\frac{x^{\beta-1}\left(1+x^\beta\right)\left\{\alpha+(\gamma-\alpha)\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\gamma\right\}}{\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]\left\{1+\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\alpha-\left[1-\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}\right]^\gamma\right\}}\right\},$$

$x>0$.

5.4. Characterization based on the conditional expectation of certain function of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X ,for $\alpha=\gamma=1$, in terms of the conditional expectation of ψ. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize EE-PL distribution.

Let $X:\Omega\to(a,b)$ be a continuous random variable with cdf F . Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+}\psi(x)=1$. Then for $\delta\neq 1$,

$$E[\psi(X)\mid X>x]=\delta\psi(x),\qquad x\in(a,b),$$

if and only if

$$\psi(x)=(1-F(x))^{\frac{1}{\delta}-1},\qquad x\in(a,b).$$

For $\alpha=\gamma=1$, $(a,b)=(0,\infty)$, $\psi(x)=\left(1+\frac{\lambda x^\beta}{1+\lambda}\right)e^{-\lambda x^\beta}$ and $\delta=\frac{\lambda}{1+\lambda}$, Proposition 5.4 provides a characterization of the EE-PL distribution.

6. Application

In this section, we illustrate the fitting performance of the EE-PL distribution using a real data set. For the purpose of comparison, we fitted the following models to show the fitting performance of EE-PL distribution by means of real data set:
i) Lindley Distribution, $L(\lambda)$.
ii) Power Lindley distribution, $PL(\beta,\lambda)$.
iii) Generalized Lindley, $GL(\alpha,\lambda)$, (Nadarajah et al. (2011)), with distribution function given by

$$F(x)=\left(1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right)^\alpha.$$

iv) Beta Lindley, $BL(\alpha,\beta,\lambda)$, with distribution function given by

$$F(x)=\int_0^{L(x,\lambda)}t^{\alpha-1}(1-t)^{\beta-1}dt.$$

v) Exponentiated power Lindley distribution, $EPL(\alpha,\beta,\lambda)$, with distribution function given by

$$F(x)=\left(1-(1+\frac{\lambda x^{\beta}}{1+\lambda})e^{-\lambda x^{\beta}}\right)^{\alpha}.$$

vi) Odd log-logistic power Lindley distribution $OLL-PL(\alpha,\beta,\lambda)$, (Alizadeh et al. (2017)), with distribution function given by

$$F(x)=\frac{PL(x,\beta,\lambda)^{\alpha}}{PL(x,\beta,\lambda)^{\alpha}+(1-PL(x,\beta,\lambda))^{\alpha}}.$$

vii) Kumaraswamy Power Lindley, $KPL(\alpha,\beta,\gamma,\lambda)$ (Broderick et al. (2012))

$$F(x)=1-[1-PL(x,\beta,\lambda)^{\alpha}]^{\gamma}.$$

viii) Odd Burr-Power Lindley, $OBu-PL(\alpha,\beta,\gamma,\lambda)$ (Altun et al.(2017a))

$$F(x)=1-\left(1-\frac{PL(x,\beta,\lambda)^{\alpha}}{PL(x,\beta,\lambda)^{\alpha}+(1-PL(x,\beta,\lambda))^{\alpha}}\right).$$

ix) Extended Exponential Lindley, $EE-L(\alpha,\gamma,\lambda)$, Ranjbar, et al. (accepted (2018)),

$$F(x)=\frac{L(x,\lambda)^{\alpha}}{L(x,\lambda)^{\alpha}+1-(1-L(x,\lambda))^{\gamma}}.$$

Estimates of the parameters of EE-PL distribution, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each dataset. We have also considered the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. Generally speaking, the smaller values of AIC, BIC, W^* and A^*, the better fit to a data set. All the computations were carried out using the software R.

Note that initial values of model parameters are quite important to obtain the correct MLEs of parameters. To avoid local minima problem, we first obtain the parameter estimate of the Lindley distribution. Then, the estimated parameter of the Lindley distribution is used as the initial value of the parameter of the PL and GL distributions. Then, the estimated parameters of PL distribution, λ and β, is used as the initial values of the EE-PL distribution. This approach is quite useful to obtain correct parameter estimates of extended models.

The data are the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958-1984 rounded to one decimal place. These data were analyzed by Akinsete et al. (2008).

The ML estimates of the parameters and the goodness-of-fit test statistics for the real data set is presented in Table 3 and 4 respectively. As we can see, the smallest values of AIC, BIC, A^*, W^* and $-l$ statistics and the largest p-values belong to the EE-PL distribution. Therefore the EE-PL distribution outperforms the other competitive considered distribution in the sense of this criteria. The OLL-L distribution provides the second best fit and this data set.

Table 2: The data set.

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3	1.4	18.7	8.5	25.5
11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0	7.3	
22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0
3.6	5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0	1.5	2.5	
27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0						

Table 3: Parameter ML estimates and their standard errors (in parentheses) for the data set.

$Model$	α	β	γ	λ
$Lindley(\lambda)$	–	–	–	0.153 (0.013)
$GL(\alpha,\lambda)$	0.508(0.076)	–	–	0.104 (0.0149)
$PL(\beta,\lambda)$	–	0.700 (0.057)	–	0.338 (0.055)
$BL(\alpha,\beta,\lambda)$	0.555(0.098)	0.274 (0.239)	–	0.333 (0.272)
$EPL(\alpha,\beta,\lambda)$	0.730(0.235)	0.915 (0.595)	–	0.300 (0.279)
$OLLPL(\alpha,\beta,\lambda)$	0.557(0.178)	1.073 (0.244)	–	0.154 (0.091)
$KPL(\alpha,\beta,\gamma,\lambda)$	1.675(2.433)	0.453 (0.432)	7.563 (11.736)	0.279 (0.522)
$OBu(\alpha,\beta,\gamma,\lambda)$	24.91(25.654)	0.024 (0.032)	41.25 (22.520)	0.984 (0.149)
$EEL(\alpha,\gamma,\lambda)$	0.618(0.101)	–	2.770 (1.704)	0.169 (0.028)
$EEPL(\alpha,\beta,\gamma,\lambda)$	4.521(3.067)	0.472 (0.094)	55.07 (58.193)	1.551 (0.643)

Table 4: Goodness-of-fit test statistics for the data set.

$Model$	AIC	BIC	$p-value$	W^*	A^*	$-l$
$Lindley(\lambda)$	530.423	532.700	0.001	0.139	0.852	264.211
$GL(\alpha,\lambda)$	509.349	513.902	0.276	0.132	0.822	252.674
$PL(\beta,\lambda)$	508.443	512.996	0.405	0.123	0.766	252.103
$BL(\alpha,\beta,\lambda)$	510.206	517.036	0.297	0.150	0.866	252.221
$EPL(\alpha,\beta,\lambda)$	510.425	517.255	0.395	0.147	0.854	252.212
$OLLPL(\alpha,\beta,\lambda)$	507.937	514.767	0.471	0.093	0.592	250.968
$KPL(\alpha,\beta,\gamma,\lambda)$	512.221	521.328	0.371	0.152	0.866	252.110
$OBu(\alpha,\beta,\gamma,\lambda)$	511.212	520.319	0.401	0.140	0.799	251.606
$EEL(\alpha,\gamma,\lambda)$	508.931	515.761	0.174	0.101	0.662	251.465
$EEPL(\alpha,\beta,\gamma,\lambda)$	500.594	509.701	0.994	0.026	0.180	246.297

In addition, the profile log-likelihood functions of the EE-PL distribution are plotted in Figure 4. These plots reveal that the likelihood equations of the EE-PL distribution have solutions that are maximizers.

Here, we also applied likelihood ratio (LR) tests. The LR tests can be used for comparing the EE-PL distribution with its sub-models. For example, the test of $H_0 : \beta = 1$ against $H_1 : \beta = 1$ is equivalent to comparing the EE-PL and EE-L distributions with each other. For this

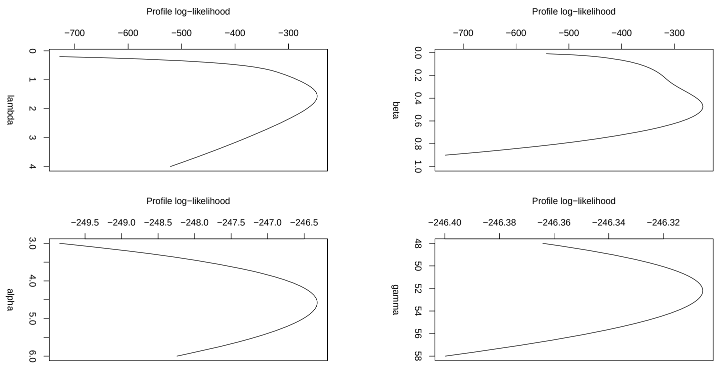

Figure 4: The profile log-likelihood functions of the EE-PL distribution.

test, the LR statistic can be calculated by the following relation

$$LR = \left[l(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\lambda}) - l(\widehat{\alpha^*}, 1, \widehat{\gamma^*}, \widehat{\lambda^*}) \right],$$

where $\widehat{\alpha^*}, \widehat{\gamma^*}$ and $\widehat{\lambda^*}$ are the ML estimators of α, γ and λ, respectively, obtained under H_0. Under the regularity conditions and if H_0 is assumed to be true, the LR test statistic converges in distribution to a chi square with r degrees of freedom, where r equals the difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, (for $H_0 : \beta = 1$, we have $r = 1$). Table 5 gives the LR statistics and the corresponding p-values.

Table 5: The LR test results.

	Hypotheses	LR	p-value
EE-PL versus Lindley	$H_0 : \alpha = \beta = \gamma = 1$	35.828	< 0.0001
EE-PL versus PL	$H_0 : \alpha = \gamma = 1$	11.612	0.0030
EE-PL versus GL	$H_0 : \alpha = \gamma,\ \beta = 1$	12.754	0.0017
EE-PL versus EPL	$H_0 : \alpha = \gamma$	11.830	0.0005
EE-PL versus EE-L	$H_0 : \beta = 1$	10.336	0.0013

From Table 5, we observe that the computed p-values are too small so we reject all the null hypotheses and conclude that the EE-PL fits the first data better than the considered sub-models according to the LR criterion.

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 4. Figure 4 suggests that the EE-PL fits the skewed data very well.

7. Conclusion

In this paper, a new distribution called Extended Exponentiated Power Lindley (EE-PL) distribution is introduced. The statistical properties of the EE-PL distribution including the hazard

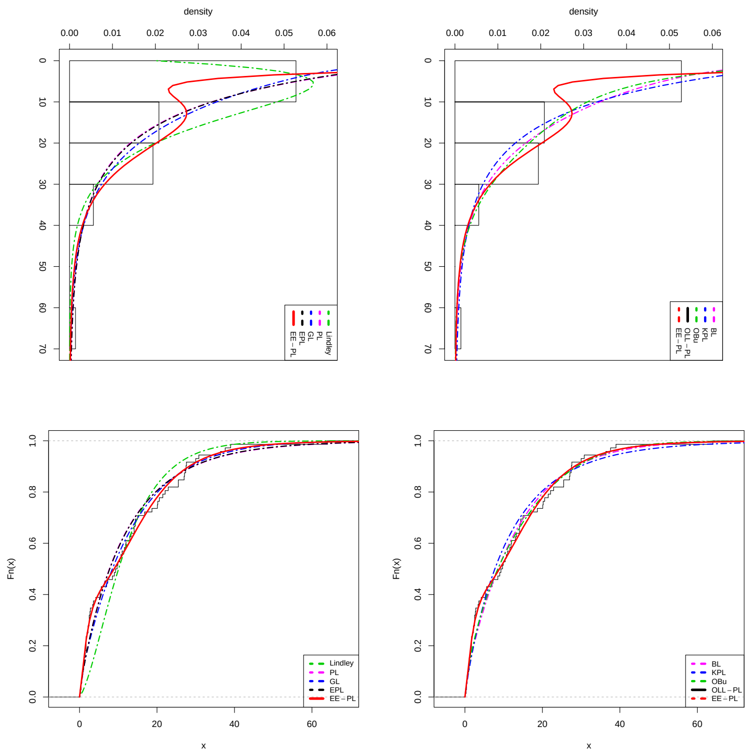

Figure 5: Fitted densities and distribution functions for the data set.

and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation for the model parameters are given. Simulation studies was conducted to examine the performance of the new EE-PL distribution. We also present applications of this new model to a real life data set in order to illustrate the usefulness of the distribution.

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