STATISTICS IN TRANSITION new series, March 2020
Vol.21, No. 1, pp. 155–162, DOI 10.21307/stattrans-2020-010
Submitted – submitted 31.12.2017; accepted for publishing – 28.11.2019

Alternative approach to moments of order statistics from one-parameter Weibull distribution

Piyush Kant Rai[1], **Anu Sirohi**[2]

ABSTRACT

The Weibull distribution is used to describe various observed failures of phenomena and widely used in survival analysis and reliability theory. Sometimes it is very difficult to compute moments of such distributions due to various reasons for e.g. analytical issues, multi parameter cases etc. This study presents the computation of the moments and the expected value of the product of order statistics in the sample from the one-parameter Weibull distribution. An alternative approach in connection to survival function is used to obtain these moments and expected values. In addition the characteristic function of the above distribution is also obtained in the form of gamma functions. Further an illustration is shown to find the first two moments and expected value of the product of order statistics by using this approach.

Key words: order statistics, survival function, moments, characteristic function.

1. Introduction

Order statistics deals with the properties and applications of ordered random variables. The concept of order statistics is widely used and play a very important role particularly in life testing experiments, in a variety of practical situations where some of the observations in the sample are censored. Since these experiments may take a long time to complete, usually it is desirable to stop after the first r failure out of n items under the test. The observations are basically the times of r failures, unlike most situations, and are obtained in order by the method of experimentation. Using the obtained data we can estimate the required parameter of interest such as true mean lifetime.

Other occurrences arise in the reliability theory and survival analysis where, a system of n components is called r out of n system if at least r components occur. For components with independent lifetime distributions $F_1, F_2, ..., F_n$ the time to failure of the system is seen to be the $(n-r+1)th$ order statistics from the set of underlying heterogeneous distributions $F_1, F_2, ..., F_n$. Several studies reveal the use of it in the characterization problems, linear estimation, detection of outliers, study of system reliability, survival analysis, life testing, data compression, among others (Arnold and Balakrishnan, 1989; Balakrishnan and Cohen, 1991). Lieblein (1953) derived a formula for variance and covariance of order statistics in sample from the extreme-value distribution (asymptotic distribution of the largest value) in terms of a certain tabulated functions. Balakrishnan and his co-author (1986) have looked

[1]Dept. of Statistics, B.H.U. Varanasi, India. E-mail: raipiyush5@gmail.com.
[2]Dept. of Mathematics and Statistics, Banasthali Vidyapith, Rajasthan, India. E-mail: bsc.ashori@gmail.com.

into the order statistics from various types of generalized logistic population. The theory of order statistics is available in the excellent book by David and Nagaraja (2003).

The order statistics from the Weibull distribution is considered in the present study. The Weibull distribution is widely used in reliability and survival analysis due to its versatility. This distribution is used to model the variety of life behaviours depending on the values of the parameters. This paper is not concerned with the properties of this distribution, a detailed account on the estimation of parameter and applications of this distribution can be seen from many studies and available researches, e.g. the books written by Prabhakar Murthy et al. (2004) and Rinne (2008). Let us consider the random samples of size n from the Weibull distribution with probability density function (pdf),

$$f(x) = \gamma x^{(\gamma-1)} e^{-x^{\gamma}}, \qquad x \geq 0, \quad \gamma > 0 \tag{1}$$

The corresponding cumulative distribution function (cdf) and survival function (sf) are $F(x) = 1 - e^{-x^{\gamma}}$ and $S(x) = e^{-x^{\gamma}}$ respectively. This distribution has been applied extensively, mainly by Weibull (Weibull, 1939a; Weibull, 1939b; Weibull, 1951; Weibull, 1952) and will be referred to by his name. Lieblein (1955), in his later paper, derived first two moments of order statistics in terms of incomplete beta and gamma function for the Weibull distribution. Sometimes the derivation is not simple and the obtained expressions are not in a very analytical form. We derive higher order moments of order statistics in the sample from one-parameter Weibull distribution using an alternative expectation formula and also their joint and characteristic functions.

2. Moments of order statistics

Let n values from one-parameter Weibull distribution after ordering in size are denoted by

$$x_1, x_2, \ldots, x_n, \qquad x_1 \leq x_2 \leq \ldots \leq x_n$$

where, $x_r, r = 1, 2, \ldots, n$ is called the rth order statistics and we seek the moments of these order statistics. The probability density function of the rth order statistics is given by

$$p(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x), \qquad -\infty < x < \infty \tag{2}$$

where, $x = x_r$. Using (1)

$$p(x) = \frac{n!}{(r-1)!(n-r)!} \gamma x^{\gamma-1} (1 - e^{-x^{\gamma}})^{(r-1)} e^{-x^{\gamma}(n-r+1)}, \qquad x \geq 0 \tag{3}$$

Now, from (3) survival function of rth order statistics is

$$S(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{r-1} (-1)^u \; {}^{r-1}C_u \; e^{-x^{\gamma}(n-r+u+1)} / (n-r+u+1) \tag{4}$$

The joint pdf of rth and sth order statistics x_r, x_s, is

$$p(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} \\ f(x)f(y), \quad -\infty < x \leq y < \infty \tag{5}$$

where $x = x_r, y = x_s, r < s, r, s = 1, 2, ..., n$. From d.f. (1) we obtain

$$p(x,y) = \frac{n!\ \gamma^2\ x^{\gamma-1}y^{\gamma-1}}{(r-1)!(s-r-1)!(n-s)!}(1-e^{-x^\gamma})^{r-1}(e^{-x^\gamma}-e^{-y^\gamma})^{s-r-1} \\ e^{-x^\gamma}\ e^{-y^\gamma(n-s+1)} \tag{6}$$

Now, joint survival function of rth and sth order statistics is

$$S(x,y) = P(X \geq x, Y \geq y)$$

$$S(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\ {}^{r-1}C_u\ {}^{s-r-1}C_v \\ e^{-x^\gamma(s-r-v+u)}\ e^{-y^\gamma(n-s+v+1)}/[(s-r-v+u)(n-s+v+1)] \tag{7}$$

The joint pdf and survival function of $x_{r_1}, x_{r_2}, ..., x_{r_k}$ order statistics are given below in (8) and (9) respectively.

$$p(x_{r_1}, x_{r_2}, ..., x_{r_k}) = \frac{n!\ \gamma^k\ (x_{r_1}^{\gamma-1}\ x_{r_2}^{\gamma-1} ...\ x_{r_k}^{\gamma-1})}{(r_1-1)!(r_2-r_1-1)!...(n-r_k)!}(1-e^{-x_{r_1}^\gamma})^{r_1-1} \\ (e^{-x_{r_1}^\gamma}-e^{-x_{r_2}^\gamma})^{r_2-r_1-1}(e^{-x_{r_2}^\gamma}-e^{-x_{r_3}^\gamma})^{r_3-r_2-1}... \\ e^{-x_{r_k}^{\gamma(n-r_k)}}\ e^{-(x_{r_1}^\gamma+x_{r_2}^\gamma+...+x_{r_k}^\gamma)} \tag{8}$$

$$S(x_{r_1}, x_{r_2}, ..., x_{r_k}) = P(X_1 \geq x_{r_1}, ..., X_k \geq x_{r_k})$$

So,

$$S(x_{r_1}, x_{r_2}, ..., x_{r_k}) = \frac{n!}{(r_1-1)!(r_2-r_1-1)!...(n-r_k)!}\sum_{u_1=0}^{r_1-1}\sum_{u_2=0}^{r_2-r_1-1}\cdots\sum_{u_{k-1}=0}^{r_{k-1}-r_{k-2}-1} \\ (-1)^{u_1+u_2+...+u_{k-1}}\ {}^{r_1-1}C_{u_1}\ {}^{r_2-r_1-1}C_{u_2}...\ {}^{r_{k-1}-r_{k-2}-1}C_{u_{k-1}} \\ \frac{e^{-x_{r_1}^\gamma(r_2-r_1-u_2+u_1)}...e^{-x_{r_{k-2}}^\gamma(r_{k-1}-r_{k-2}-u_{k-1}+u_{k-2})}e^{-x_{r_{k-1}}^\gamma(u_{k-1}+1)}e^{-x_{r_k}^\gamma(n-r_k+1)}}{(r_2-r_1-u_2+u_1)...(r_{k-1}-r_{k-2}-u_{k-1}+u_{k-2})(u_{k-1}+1)(n-r_k+1)} \tag{9}$$

where $r_1, r_2, ..., r_k = 1, 2, ..., n$, such that $r_1 < r_2 < ... < r_k$. This note utilizes survival function to find the moments of order statistics from one-parameter Weibull distribution. Recently, this technique of finding moments has gained more importance in practical situation and Hong called it an alternative expectation formula (Feller, 1966; Nadarajah and Mitov, 2003; Hong, 2012). For a continuous non-negative random variable X, the mean or the first moment can be expressed as

$$E(X) = \int_0^{\infty} (1 - F(x))dx = \int_0^{\infty} S(x)dx \tag{10}$$

Hong (2012) derived this formula to find the mean. Later he established the formula for the expected value of the joint variables (Hong, 2015). If $X_1, X_2, ..., X_n$ are non-negative random variables and $S(x_1, x_2, ..., x_n)$ is their joint survival function then expected value of joint variables (10) is

$$E(X_1, X_2, ..., X_n) = \int_0^{\infty} ... \int_0^{\infty} S(x_1, x_2, ..., x_n)dx_1 dx_2 ... dx_n \tag{11}$$

Chakraborti et al. (2017) described higher order moments using the same technique. The mth moments about the origin, $E(X^m)$, of a continuous random variable X is

$$E(X^m) = \int_0^{\infty} m x^{m-1} S(x)dx \tag{12}$$

From the survival function (4) of rth order statistics we obtain

$$E(X_r^m) = \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{r-1} \frac{(-1)^u \ {}^{r-1}C_u}{(n-r+u+1)} \int_0^{\infty} m x^{m-1} e^{-x^{\gamma}(n-r+u+1)} dx \tag{13}$$

Making the transformation $x^{\gamma} = z$ in (13) and performing some algebraic simplification we obtain

$$E(X_r^m) = \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{r-1} (-1)^u \ {}^{r-1}C_u \frac{(m/\gamma)\Gamma(m/\gamma)}{(n-r+u+1)^{m/\gamma+1}} \tag{14}$$

From (14) we can find moments about origin of the order statistics from the Weibull distribution. Combining (11) and (7) we obtain

$$E(X,Y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \frac{{}^{r-1}C_u \ {}^{s-r-1}C_v}{(n-s+v+1)(s-r-v+u)} \int_0^{\infty} \int_0^{y} e^{-x^{\gamma}(s-r-v+u)} \ e^{-y^{\gamma}(n-s+v+1)} dxdy \tag{15}$$

Making the transformation $x^{\gamma} = z$ and performing some algebra we get

$$E(X,Y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \frac{{}^{r-1}C_u \ {}^{s-r-1}C_v}{\gamma(n-s+v+1)(s-r-v+u)} \int_0^{\infty} G(1/\gamma, y^{\gamma}) \ e^{-y^{\gamma}(n-s+v+1)} dy \tag{16}$$

where,

$$G(1/\gamma, y^\gamma) = \int_0^{y^\gamma} e^{-z(s-r-v+u)} z^{(1/\gamma)-1} dz \tag{17}$$

On multiplying $e^{y^\gamma(s-r-v+u)}$ both sides and making transformation $z = y^\gamma t$, and performing some algebra in (17) we get

$$G(1/\gamma, y^\gamma) = \sum_{w=0}^{\infty} \frac{(s-r-v+u)^w}{w!} B(1/\gamma, w+1)(y^\gamma)^{(1/\gamma)+w} e^{-y^\gamma(s-r-v+u)} \tag{18}$$

where $B(1/\gamma, w+1)$ is the beta function. Combining (16) and (18) and making another transformation $y^\gamma = p$ and performing some algebraic simplification we get

$$E(X,Y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{u=0}^{r-1} \sum_{v=0}^{s-r-1} (-1)^{u+v} \frac{{}^{r-1}C_u \; {}^{s-r-1}C_v}{\gamma^2(n-s+v+1)(s-r-v+u)}$$
$$\sum_{w=0}^{\infty} \frac{(s-r-v+u)^w}{w!(n-r+u+1)^{(2/\gamma+w)}} B(1/\gamma, w+1)\Gamma((2/\gamma)+w) \tag{19}$$

In the same manner the expected value of the product of order statistics $x_{r_1}, x_{r_2}, ..., x_{r_k}$ from the Weibull distribution is

$$E(X_{r_1}, X_{r_2}, ..., X_{r_k}) = \int_0^\infty \int_0^{x_{r_k}} \int_0^{x_{r_{k-1}}} ... \int_0^{x_{r_2}} S(x_{r_1}, x_{r_2}, ..., x_{r_k}) \, dx_{r_1} dx_{r_2} ... dx_{r_k} \tag{20}$$

After simplifications,

$$E(X_{r_1}, X_{r_2}, ..., X_{r_k}) = \frac{n!}{\gamma^k (r_1-1)!(r_2-r_1-1)!...(n-r_k)!} \sum_{u_1=0}^{r_1-1} \sum_{u_2=0}^{r_2-r_1-1} ... \sum_{u_{k-1}=0}^{r_{k-1}-r_{k-2}-1}$$
$$(-1)^{u_1+u_2+...+u_{k-1}} \; {}^{r_1-1}C_{u_1} \; {}^{r_2-r_1-1}C_{u_2} ... {}^{r_{k-1}-r_{k-2}-1}C_{u_{k-1}}$$
$$\left(\sum_{w_1=0}^{\infty} \frac{(r_2-r_1-u_2+u_1)^{w_1} B(1/\gamma, w_1+1)}{w_1!(r_2-r_1-u_2+u_1)}\right) ...$$
$$\left(\sum_{w_{k-2}=0}^{\infty} \frac{(r_{k-2}-r_1-u_{k-2}+u_1)^{w_{k-2}} B((k-2/\gamma)+w_1+...+w_{k-3}, w_{k-2}+1)}{w_{k-2}!(r_{k-2}-r_1-u_{k-2}+u_1)}\right) \tag{21}$$
$$\left(\sum_{w_{k-1}=0}^{\infty} \frac{(r_{k-1}-r_1-u_{k-1}+u_1)^{w_{k-1}} B((k-1/\gamma)+w_1+...+w_{k-2}, w_{k-1}+1)}{w_{k-1}!(u_{k-1}+1)}\right)$$
$$\frac{\Gamma((k/\gamma)+w_1+...+w_{k-1})}{(n-r_k+1)(n-r_k+r_{k-1}-r_1+u_1+2)^{(k/\gamma)+w_1+...+w_{k-1}}}$$

3. characteristic function of order statistics

The characteristic function of rth order statistics x_r is given as

$$\phi_r(t) = E(e^{itx}) \tag{22}$$

where, $x = x_r$, from (2) we obtain

$$\phi_r(t) = \frac{n!}{(r-1)!(n-r)!} \int_0^{\infty} e^{itx} \gamma x^{\gamma-1} (1 - e^{-x^{\gamma}})^{(r-1)} e^{-x^{\gamma}(n-r+1)}$$

$$= \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{r-1} (-1)^{u\, r_1-1} C_{u_1} \int_0^{\infty} \gamma x^{\gamma-1} e^{-x^{\gamma} u} e^{itx} e^{-x^{\gamma}(n-r+1)} \tag{23}$$

On expanding e^{itx} in terms of power series, making transformation $x^{\gamma} = z$ and performing some algebra in (20) we get

$$\phi_r(t) = \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{r-1} (-1)^{u\, r_1-1} C_{u_1} \sum_{p=0}^{\infty} \frac{(it)^p}{p!} \frac{p/\gamma \Gamma(p/\gamma)}{(n-r+u+1)^{p/\gamma+1}} \tag{24}$$

4. Illustration

The above formulae are used to find the first two moments for a small sample of one-parameter Weibull distribution by means of order statistics. The following computations for $n = 3$ and $\gamma = 2.5$ illustrate the first and second moments corresponding to $r = 1, r = 2$ and $r = 3$,

$$E(X_1) = 0.572,\ E(X_2) = 0.874$$

$$E(X_3) = 1.216$$

and

$$E(X_1^2) = 0.387,\ E(X_2^2) = 0.831$$

$$E(X_3^2) = 1.575$$

Variance corresponding to these statistics for $r = 1, r = 2$ and $r = 3$ is,

$$V(X_1) = 0.572,\ V(X_2) = 0.874$$

$$V(X_3) = 1.216$$

The expected value of joint of order statistics for $r = 1$ and $s = 2$ is,

$$E(X,Y) = 0.728$$

In the same manner we can find out the higher order moments about origin and mean on the basis of the procedure described in this article.

5. Conclusion

The Weibull distribution is used to describe various observed failures of phenomena and widely used in survival analysis and reliability theory. Techniques introduced here to evaluate the moments of order statistics from one-parameter Weibull distribution involve survival function and will be highly useful for practical situations. Characteristic function has great ttheoretical importance and it has central importance in statistics and the expression is derived here for the order statistics from Weibull distribution. Moreover it is able to generate higher order moments of order statistics and can be used to evaluate the expression for mean, variance, covariance, skewness and kurtosis of the distribution.

REFERENCES

ARNOLD, B. C., BALAKRISHNAN, N., (1989). Relations, bounds and applications for order statistics, Springer-Verlag, lecture Notes in Statistics, No. 53.

BALAKRISHNAN, N., COHEN, A. C., (1991). Order statistics and inference: Estimation methods, Academic Press.

BALAKRISHNAN, N., KOCHERLAKOTA, (1986). On the moments of order statistics from the doubly truncated logistic distribution, Journal of Statistical Planning and Inference, 13, pp.117–129.

CHAKRABORTI, S., JARDIM, F. AND EPPRECHT, E., (2017). Higher order moments using the survival function: The alternative expectation formula, The American Statistician, pp. 1–12.

DAVID H.A., NAGARAJA H. N., (2003). Order Statistics, Third Edition, John Wiley, New York.

FELLER, W., (1966). An introduction to probability theory and its application, New York: Wiley.

HONG, L., (2012). A remark non the alternative expectation formula, The American Statistician, 66, pp. 232–233.

HONG, L., (2015). Another remark on the alternative expectation formula, The American Statistician, 69(3), pp. 232–233.

LIEBLEIN, J., (1953). On the exact evaluation of the variances and covariances of order statistics in samples from the extreme-value distribution, Annals of Mathematical Statistics, 24, pp. 282–287.

LIEBLEIN, J., (1955). On moments of order statistics from Weibull distribution. Annals of Mathematical Statistics, 24, pp. 330–333.

NADARAJAH, S., MITOV, K. V., (2003). Product moments of multivariate random vectors, Communication in Statistics: Theory and Methods, 32(1), pp. 47–60.

PRABHAKAR MURTHY, D. N., XIE, M., JLANG, R., (2004). The Weibull Model. Weily Series in Probability and Statistics.

RINNE, H., (2008). The Weibull distribution, CRC Press.

WEIBULL, W., (1939a). A statistical theory of the strength and materials, Ing. Vetenskaps Akad. Handl., 151, p. 15.

WEIBULL, W., (1939b). The Phenomenon of rupture in solids. Ing. Vetenskaps Akad. Handl., 153, p. 17.

WEIBULL, W., (1951). A statistical distribution of wide applicability. J. Appl. Mech., 18, pp. 293–297.

WEIBULL, W., (1952). Statistical design of fatigue experiments. J. Appl. Mech., 19, pp. 109–113.