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Beta transmuted Lomax distribution with applications

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ABSTRACT

In this paper we propose and test a composite generalizer of the Lomax distribution .The genesis of the beta distribution and transmuted map is used to develop the so-called beta transmuted Lomax (BTL) distribution. The properties of the distribution are discussed and explicit expressions are derived for the moments, mean deviations, quantiles, distribution of order statistics and reliability. The maximum likelihood method is used for estimating the model parameters, and the finite sample performance of the estimators is assessed by simulation. Finally, the authors demonstrate the usefulness of the new distribution in analysing positive data.

Key words: Lomax distribution, beta Lomax distribution, transmuted distribution, maximum likelihood estimation.

1. Introduction

Lomax (1954) proposed Pareto Type – II (the shifted Pareto) distribution, also known as Lomax distribution, and used it for the analysis of business failure lifetime data. The Lomax distribution is widely applicable in reliability and life testing problems in engineering as well as in survival analysis as an alternative distribution. After the work of Lomax (1954), various authors studied the Lomax distribution.

The cumulative distribution function of the Lomax arises as a limit distribution for the residual lifetime at a great age (Balkema and De Haan, 1974). Myhre and Saunders (1982) gave application of the Lomax distribution using the right censored data. Lingappaiah (1986) proposed various procedures of estimation for the Lomax distributions. Nayak (1987) proposed a multivariate Lomax distribution and discussed its various properties and usefulness in reliability theory. Ahsanullah (1991) and Balakrishnan and Ahsanullah (1994) investigated distributional properties and moments of record values from the Lomax distribution respectively. Vidondo et al. (1997) used this distribution for modelling size spectra data in aquatic ecology.

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Childs et al. (2001) gave order statistics from nonidentical right-truncated Lomax distributions and gave applications for these situations. The Lomax distribution was used to obtain a discrete Poisson Lomax distribution (Al-Awadhi and Ghitany, 2001). Bayesian method of estimation was used for the estimation of the Lomax survival function (Howlader and Hossain, 2002). Non-Bayesian and Byesian estimators of the sample size in the case of type I censored samples from the Lomax distribution were proposed by Abd-Elfattah et al. (2007). Ghitany et al. (2007) proposed the properties of a new parametric distribution, which was investigated by Marshall and Olkin (1997) and comprehensively extended the distributions family which is being applied to the model of the Lomax. Hassan and Al-Ghamdi (2009) used the Lomax distribution for the determination of optimal times of changing level of stress for simple stress plans under a cumulative exposure model. Abd-Elfattah and Alharbey (2010) estimated the parameters of the Lomax distribution based on generalized probability weighted moments. Abdul-Moniem and Abdel-Hameed (2012) studied exponentiated Lomax (EL), Abdullahi and Ieren (2018) introduced transmuted Exponential Lomax distribution (TEL), Ghitany et al. (2007) introduced Marshall-Olkin extended Lomax (MOEL), Bindu and Sangita (2015) studied double Lomax (DL).
The probability density function (pdf) and the cumulative distribution function (cdf) of a Lomax distribution is given by

$$g(x) = \gamma\theta(1+\gamma x)^{-\theta-1}, \;\; x > 0, \; \gamma, \theta > 0, \tag{1}$$

$$G(x) = \left[1-(1+\gamma x)^{-\theta}\right], \;\; x > 0, \; \gamma, \theta > 0. \tag{2}$$

In this article we propose a new generalizer, which is obtained by the composition of the genesis of beta distribution and transmutation map. We will execute this generalizer to the Lomax distribution to develop the so-called beta transmuted Lomax distribution. This will be the beta generalizer of the transmuted Lomax (TL) distribution studied by Ashour and Eltehiwy (2013). Consider a baseline cumulative distribution function (cdf) $G(x)$ with corresponding probability density function (pdf) $g(x)$ and parameter vector Θ. Then, the cdf of the transmuted Lomax (TL) family of distributions (for $x > 0$) is

$$\mathrm{G(x)} = \left(1-(1+\gamma x)^{-\theta}\right) \; \left(1+\lambda\,(1+\gamma x)^{-\theta}\right). \tag{3}$$

The corresponding pdf of the transmuted Lomax distribution is given by

$$\mathrm{g(x)} = \frac{\gamma\theta}{(1+\gamma x)^{\theta+1}} \; \left(1-\lambda+2\lambda(1+\gamma x)^{-\theta}\right), \tag{4}$$

where $|\lambda| \leq 1$.

A class of generalized distributions F(x) has received considerable attention over the last few years, in particular, after the studies by Eugene, Lee, and Famoye (2002) and Jones (2004). If G denotes the baseline cumulative distribution function (cdf) of a random variable, then the beta generalized distribution is defined as

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1} (1-t)^{b-1} dt \,, \tag{5}$$

where $a > 0$ and $b > 0$ are shape parameters. Note that $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the incomplete beta function ratio, and $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1} \; dt$ is the incomplete beta function, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and $\Gamma(.)$ is the gamma function. The probability density function (pdf) of the Beta generated distribution has the form

$$\mathrm{f(x)} = \frac{\mathrm{g(x)}}{\mathrm{B}(a,\mathrm{b})} \, [\mathrm{G}(x)]^{\mathrm{a}-1}[1-\mathrm{G}(x)]^{\mathrm{b}-1}. \tag{6}$$

Based on the above generalization, Lemonte and Cordeiro (2013) investigated beta Lomax (BL), Gupta et al. (2016) introduced Lomax-Gumbel and studied expressions for its characteristic function, Terna and David (2018) introduced the new extension of the exponential distribution is called Lomax-Exponential distribution (LED), Kawsar et al. (2018) introduced Rayleigh Lomax distribution, Kumaraswamy Lomax (KwL) and McDonald Lomax (McL) and Cordeiro et al. (2013) introduced gamma Lomax (GL) distributions. Recently Tahir et al. (2015) introduced the Weibull Lomax (WL) distribution and studied its mathematical and statistical properties, Tahir et al. (2016) introduced the Gumbel Lomax (GL) distribution and Oguntunde et al. (2018) developed a new compound distributions which is Gompertz Lomax distribution.

The main aim of this paper is to define and study a new family of distributions by adding two extra shape parameters in (3) to provide more flexibility to the generated family. The additional advantage of the new distribution is that it has more parameters to have a better control. The rest of the paper is organized as follows. In Section 2 we define the BTL distribution and discuss some of its sub-models. In Section 3 we present the mixture representation of the BTL distribution. Section 4 discusses mathematical characteristics of the BTL distribution such as the moments, quantile, mean deviation, order statistics and stress-strength model. Estimation of parameters by the maximum likelihood method and the performance of the estimators is assessed by simulation in Section 5. In Section 6, the distribution is used for analysing real data. Finally, in Section 7, we make some concluding remarks on our study.

2. The beta transmuted Lomax distribution

In this section we provide the formulation of the beta transmuted Lomax (BTL) distribution. By inserting (3) into (5) the cumulative distribution function of the beta-transmuted Lomax distribution with five parameters is given by

$$F(x) = I_{\left(1-(1+\gamma x)^{-\theta}\right)\ \left(1+\lambda(1+\gamma x)^{-\theta}\right)} \quad (a,b)$$

$$= \frac{1}{B(a,b)} \int_0^{(1-(1+\gamma x)^{-\theta})\ (1+\lambda(1+\gamma x)^{-\theta})} t^{\alpha-1}(1-t)^{b-1} dt, \quad (7)$$

where $x > 0,\ \gamma, \theta > 0, |\lambda| \leq 1$ and $a > 0, b > 0$.

The cdf can be expressed in a closed form using the hypergeometric function (see Cordeiro and Nadarajah 2011) as follows:

$$F(x) = \frac{[(1-(1+\gamma x)^{-\theta})\ (1+\lambda(1+\gamma x)^{-\theta})]^a}{aB(a,b)} . 2F1\big(a, 1-b; a+1; \big(1-(1+\gamma x)^{-\theta}\big)\ \big(1+\lambda(1+\gamma x)^{-\theta}\big)\big),$$

where $2F1(c,d;e;z) = \sum_{k=0}^{\infty} \frac{(c)_k\,(d)_k}{(e)_k} \frac{(z^k)}{k!}$ is the Gaussian hypergeometric function, where $(c)_k$ is the ascending factorial defined by (assuming that $(c)_0 = 1$)

$$(c)_k = \begin{cases} c(c+1)(c+2) \ldots .(c+k-1) & k = 1,2,3, \ldots \\ 1 & k = 0 \end{cases}$$

Differentiating (7) with respect to x, we get the probability density function of the BTL distribution given by

$$\mathrm{f(x)} = \frac{\gamma\theta}{\mathrm{B(a,b)}} (1+\gamma x)^{-\theta-1}\big(1-\lambda+2\lambda(1+\gamma x)^{-\theta}\big) \big[\big(1-(1+\gamma x)^{-\theta}\big)\big]^{\mathrm{a}-1} \big[\big(1+\lambda(1+\gamma x)^{-\theta}\big)\big]^{\mathrm{a}-1} \big[1-\big(1-(1+\gamma x)^{-\theta}\big)\big(1+\lambda(1+\gamma x)^{-\theta}\big)\big]^{\mathrm{b}-1}, \quad (8)$$

where $x > 0,\ \gamma, \theta > 0, |\lambda| \leq 1$ and $a > 0, b > 0$.

The beta transmuted Lomax (BTL) distribution includes the following distributions as a special case:

- for $\lambda = 0$, beta transmuted Lomax reduces to beta Lomax distribution.
- For $\mathrm{a} = \mathrm{b} = 1$, beta transmuted Lomax reduces to transmuted Lomax distribution.
- For $\lambda = 0$ and $\mathrm{b} = 1$, beta transmuted Lomax reduces to exponentiated Lomax distribution.
- For $\mathrm{a} = \mathrm{b} = 1$ and $\lambda = 0$, beta transmuted Lomax reduces to Lomax distribution.

Figure 1 illustrates some of the possible shapes of the density function of the BTL distribution for selected values of the parameters θ, λ, a and b with $\gamma = 1$.

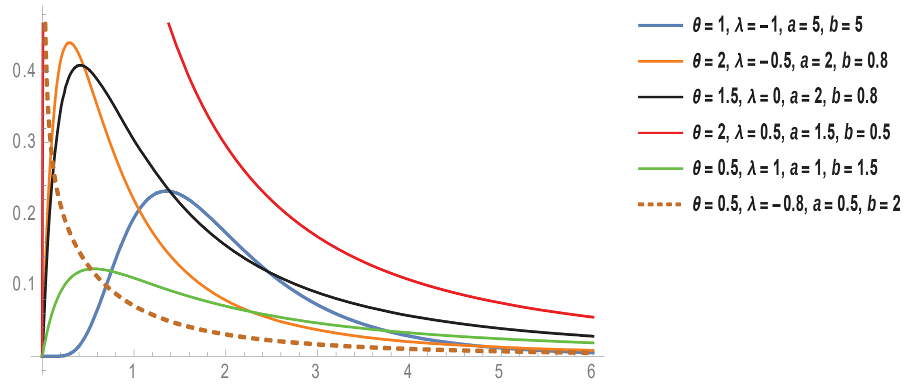

Figure 1. pdf of the BTL distribution for selected values of the parameters

The plot for PDF reveals that the BTL distribution is positively skewed and therefore will be a good model for positively skewed data sets.

Figure 2 illustrates some of the possible shapes of the cumulative distribution function of the BTL distribution for selected values of the parameters θ, λ, a and b with $\gamma = 1$.

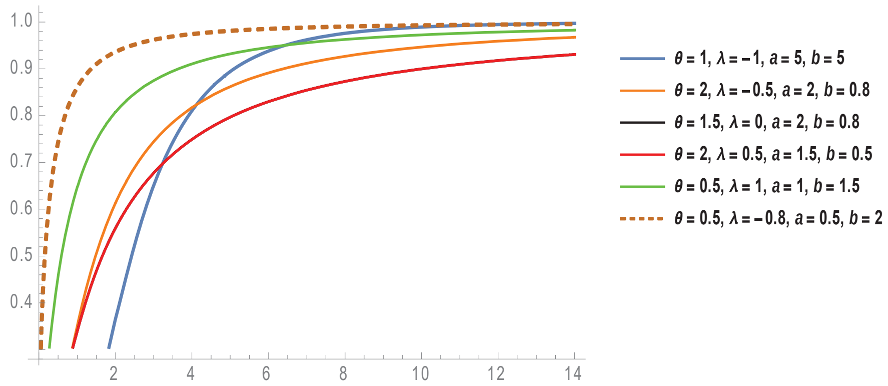

Figure 2. cdf of the BTL distribution for selected values of the parameters

The graphical representation of the cumulative function for different possible values of the parameters is shown in Figure 2, which is always an increasing function.

3. Mixture representation

In this section we find the series representations of cdf and pdf of the BTL distribution, which will be useful for studying its mathematical characteristics. As we shall see both pdf and cdf of BTL distribution can be expressed in terms of the Lomax distribution. By using (5) and the power series expansion of $(1-t)^{b-1}$, we get

$$F(x)=\frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}\,(1-t)^{b-1}dt=\frac{1}{B(a,b)}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{[G(x)]^{a+j}}{(a+j)}$$

with the binomial term $\binom{b-1}{j}=\frac{\Gamma(b)}{\Gamma(b-j)\,j!}$ defined for any real b. Hence, (7) reduces to

$$F(x)=\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{\left[\left(1-(1+\gamma x)^{-\theta}\right)\left(1+\lambda\,(1+\gamma x)^{-\theta}\right)\right]^{a+j}}{B(a,b)(a+j)}. \qquad (9)$$

Again, using the binomial expansion of $\left[\left(1-(1+\gamma x)^{-\theta}\right)\left(1+\lambda\,(1+\gamma x)^{-\theta}\right)\right]^{a+j}$, we have

$$F(x)=\sum_{j,k,l=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\lambda^l\frac{(1+\gamma x)^{-\theta(k+l)}}{B(a,b)(a+j)}$$

$$=\sum_{j,k,l=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\lambda^l\frac{\left(1-G_1\left(x;\gamma,\theta(k+l)\right)\right)}{B(a,b)(a+j)}, \qquad (10)$$

where $G_1\left(x;\gamma,\theta(k+l)\right)$ is the Lomax cdf with scale γ and shape $\theta(k+l)$ parameter. Differentiating (10) with respect to x gives a useful expansion of $f(x)$ as

$$f(x)=\sum_{k,l=0}^{\infty}w_{kl}\;g\left(x;\gamma,\theta(k+l)\right),\qquad x>0, \qquad (11)$$

where

$$w_{kl}=\sum_{j=0}^{\infty}(-1)^{j+k+l}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\frac{\lambda^l}{B(a,b)(a+j)}$$

and $g\left(x;\gamma,\theta(k+l)\right)$ is the Lomax pdf with scale γ and shape $\theta(k+l)$ parameters. If $b>0$ is an integer, the index j in the sum stops at $b-1$, and if a is an integer, then the indices k and l in the sum stop at $a+j$.

Thus, several mathematical properties of the BTL distribution can be obtained simply from those properties of the exp-G family. Equations (10) and (11) are the main result of this section.

4. Mathematical characteristics

In this section we provide some mathematical properties of the BTL distribution including the moments and moment generating function, quantiles, mean deviations, order statistics and stress-strength model.

4.1. Moments and moments generating

Moments are necessary and important in any statistical analysis, especially in applications. They can be used to study the most important features and characteristics of a distribution (e.g. tendency, dispersion, skewness and kurtosis). Using the mixture representation described in Section 3, the r-*th* moment of the BTL random variable X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r \, \mathrm{f}(x)\mathrm{dx} = \int_0^{\infty} x^r \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \; f\big(x;\gamma,\theta(k+l)\big)\,\mathrm{dx}$$

$$= \int_0^{\infty} \sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\, x^r \;\; \gamma\theta\,(k+l)(1+\gamma x)^{-\theta(k+l)-1}\mathrm{dx}$$

$$= \gamma^{-r} \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \frac{\Gamma[1+r]\Gamma[-r+(k+l)\theta]}{\Gamma[(k+l)\theta]}, \;\; (k+l)\theta > r, \tag{12}$$

$$E(X) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[\frac{1}{\gamma[-1+(k+l)\theta]}\right], \;\; (k+l)\theta > 1, \tag{13}$$

$$E(X^2) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[\frac{2(k+l)\theta\Gamma[-2+(k+l)\theta]}{\gamma^2[1+(k+l)\theta]}\right], \;\; (k+l)\theta > 2,$$

$$E(X^3) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[\frac{6(k+l)\theta\Gamma[-3+(k+l)\theta]}{\gamma^3[1+(k+l)\theta]}\right], \;\; (k+l)\theta > 3,$$

$$E(X^4) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[\frac{24(k+l)\theta\Gamma[-4+(k+l)\theta]}{\gamma^4[1+(k+l)\theta]}\right], (k+l)\theta > 4.$$

The variance, skewness and kurtosis measures can now be calculated using the relations

$$Var(x) = E(x^2) - [E(x)]^2 = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left(\frac{(k+l)\theta}{\gamma^2(-2+(k+l)\theta)(-1+(k+l)\theta)^2}\right), \tag{14}$$

$$Skewness(x) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[\frac{2+2(k+l)\theta}{\gamma(-3(k+l)\theta)\sqrt{\frac{(k+l)\theta}{\gamma^2(-2+(k+l)\theta)(-1+(k+l)\theta)^2}}\;(-1+(k+l)\theta)}\right], \tag{15}$$

$$Kurtosis(x) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left[9 - \frac{1}{(k+l)\theta} + \frac{81}{-4+(k+l)\theta} - \frac{32}{-3+(k+l)\theta}\right]. \tag{16}$$

Similarly, the moment generating function of X may be obtained as below:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left(e^{-\frac{t}{\gamma}} (k+l)\theta\, E_{\theta(k+l)+1}\left(-\frac{t}{\gamma}\right)\right), \tag{17}$$

where $E_n(z) = \int_1^{\infty} \frac{e^{-tz}}{t^n}\, dt$.

4.2. Quantiles

Quantiles are the points in a distribution that relate to the rank order of values. The quantile function of a distribution is the real solution of $F(x_q) = q$ for $0 \le q \le 1$. The quantiles of beta transmuted Lomax distribution are obtained from (7) as

$$X = \frac{\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda\left(I_q^{-1}(a,b)\right)}}{2\lambda}\right]^{-\frac{1}{\theta}} - 1}{\gamma} \tag{18}$$

where $I_q^{-1}(a, b)$ is the inverse of the incomplete beta function with parameters a and b. The following expansion for the inverse of the beta incomplete function $I_q^{-1}(a, b)$ can be found on the Wolfram website http://functions.wolfram.com/06.23.06.0004.01

$$\mathrm{I_u^{-1}(a,b)} = \mathrm{w} + \frac{\mathrm{b-1}}{\mathrm{a+1}}\mathrm{w}^2 + \frac{\mathrm{(b-1)(a^2+3ab-a+5b-4)}}{\mathrm{2(a+1)^2(a+2)}}\mathrm{w}^3 + \frac{\mathrm{(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2]}}{\mathrm{2(a+1)^2(a+2)}}\mathrm{w}^4 + \frac{\mathrm{(b-1)[(33a^2-30b+4)a+b(31a-47)+18]}}{\mathrm{3(a+1)^3(a+2)(a+3)}}\mathrm{w}^4 + \mathrm{O}\left(\mathrm{P}^{\frac{5}{\mathrm{a}}}\right),$$

where $w = \{\mathrm{a}\mathrm{B}(a, \mathrm{b}\,)q\}^{\frac{1}{a}}, a > 0$.

4.3. Mean deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has a BTL distribution, then we can derive the mean deviations about the mean $\mu = E(X)$ and about the median M as

$$\delta_1(x) = \int_0^{\infty} |x-\mu| f(x) dx,$$

and

$$\delta_2(x) = \int_0^{\infty} |x-M| f(x) dx.$$

The mean of the distribution is obtained from (12), and the median is obtained by solving the equation

$$I_{(1-(1+\gamma x)^{-\theta})\left(1+\lambda-\lambda(1-(1+\gamma x)^{-\theta})\right)}(a,b) = \frac{1}{2}.$$

These measures can be calculated using the relationships that

$$\delta_1(x) = \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x - \mu) f(x) dx$$

$$= 2 \int_0^{\mu} (\mu - x) f(x) dx$$

$$= 2\{\mu F(\mu) - \textstyle\int_0^{\mu} x f(x) dx\}$$

$$\delta_1(x) = 2\{\mu F(\mu) - J(\mu)\},$$

and

$$\delta_2(x) = \mu - 2\,J(\mu),$$

where $J(t) = \int_0^t x f(x) dx$. From (11) we have

$$J(t) = \sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \int_0^t (k + l)\theta\gamma\ x\,(1 + \gamma x)^{-\theta(k+l)-1}\, dx$$

$$= \textstyle\sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \left(\frac{(1+\gamma t)^{-\theta(k+l)}\left(-1+(1+\gamma t)^{-\theta(k+l)}-(k+l)t\gamma\theta\right)}{\gamma(-1+(k+l)\theta)} \right). \qquad (19)$$

Using (10), one can easily find $\delta_1(x)$ and $\delta_2(x)$.

4.4. Order statistics

Let $X_1, X_2, X_3, \dots\dots, X_n$ be a simple random sample from the BTL distribution with cumulative distribution function (7) and probability density function (8).

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics from this sample. The pdf $f_{i:n}(x)$ of i-th order statistics $X_{(i)}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n - i + 1)} f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i}, \quad i = 1,2, \dots, n$$

and cdf is given by

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}$$

$$= \textstyle\int_0^{F(x)} \frac{1}{B(i,n-i+1)}\ t^{i-1}[1 - t]^{n-i} dt.$$

The pdf of the j^{th} order statistic for the beta transmuted Lomax distribution is given by

$$f_{i:n}(x) = \frac{1}{B(i, n - i + 1)}\ f(x) \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} [F(x)]^{i+s-1}$$

$$f_{i:n}(x) = \frac{\theta\gamma}{\mathrm{B}(\mathrm{i}, \mathrm{n}-\mathrm{i}+1)} \left(\sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}}\,(k+l)(1+\gamma x)^{-\theta(k+l)-1} \right)$$
$$\sum_{\mathrm{s}=0}^{\mathrm{n-i}} (-1)^{\mathrm{s}+1} \binom{\mathrm{n}-\mathrm{i}}{\mathrm{s}} \left[\sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}} \left(1-(1+\gamma x)^{-\theta(k+l)}\right) \right]^{\mathrm{i+s-1}}. \quad (20)$$

Writing $= (1+\gamma x)^{-\theta}$, $f_{i:n}(x)$ can be expressed as

$$f_{i:n}(x) = \frac{\theta\gamma}{\mathrm{B}(\mathrm{i,n-i+1})} \left(\sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}}\ (\mathrm{k}+\mathrm{l})\ \mathrm{u}^{(\mathrm{k+l})-1} \right) \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left[\sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\ u^{(k+l)} \right]^{i+s-1}. \quad (21)$$

We note that in (21) we can write

$$\sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\, u^{(k+l)} = \sum_{m=0}^{\infty} w_m^*\, u^m$$

and

$$\sum_{k,l=0}^{\infty} \mathrm{w}_{kl}\ (k+l)\, u^{(k+l)} = \sum_{m=0}^{\infty} m\, w_m^*\, u^m,$$

where $w_m^* \sum_{k,l:k+l=m} \mathrm{w}_{kl}$. Further, from Gradshteyn and Ryzhik (2000), for any positive integer r,

$$\left(\sum_{k=0}^{\infty} \mathrm{a}_k\, u^k\right)^r = \sum_{k=0}^{\infty} d_{r,k}\, u^k, \quad (22)$$

where the coefficients $d_{r,k}$, for $k = 1,2,..,$ can be determined from the recurrence equation

$$d_{r,k} = \left(k_{a_0}\right)^{-1} \sum_{m=1}^{k} [m(r+1)-k]\, \mathrm{a}_m d_{r,k-m} \quad (23)$$

and $d_{r,0} = a_0^r$. Hence, $d_{r,k}$ comes directly from $d_{r,0}, \dots, d_{r,k-1}$ and, therefore, from $\mathrm{a}_0, \dots, \mathrm{a}_k$. Using (22) and (23) it follows that

$$f_{i:n}(x) \frac{\theta\gamma}{B(i,n-i+1)} \left(\sum_{m=0}^{\infty} m\, w_m^*\, u^m\right) \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left(\sum_{m=0}^{\infty} \mathrm{d}_{i+s-1,m^{u^m}}\right),$$

where

$$\mathrm{d}_{i+s-1,m} = (m\, w_0^*)^{-1} \sum_{q=1}^{k} [q(i+s)-m] w_m^* \mathrm{d}_{i+s-1,m-q},$$
$$\mathrm{d}_{i+s-1,0} = (w_0^*)^{i+s-1} = \left(\sum_{j=0}^{\infty} (-1)^{j+1} \binom{b-1}{j} \frac{1}{B(a,b)(a+j)} \right)^{i+s-1}.$$

Combining terms, we obtain

$$f_{i:n}(x) = \frac{\theta\gamma}{B(i,n-i+1)} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} m\mathrm{d}_{i+s-1,t^{w^*}t^{w^*}m^{u^{m+t}}}$$

$$= \frac{1}{B(i,n-i+1)} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} \frac{m d_{i+s-1,t} w_m^*}{m+t} \left[\theta\gamma(m + t)\,(1+\gamma x)^{-\theta(m+t)-1}\right]$$

$$= \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} c_i(m,t)\ g\big(x{:}\,\gamma, \theta(m+t)\big)\,, \quad (24)$$

where $g\big(x{:}\,\gamma, \theta(m+t)\big)$ denotes the pdf of a Lomax distribution with parameter $(m+t)\theta$ and γ parameter and

$$c_i(m,t) = \frac{1}{B(i,n-i+1)} \frac{m\, w_m^*}{m+t} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} d_{i+s-1,t}. \quad (25)$$

4.5. Stress-strength model

A stress-strength model describes the life of a component which has a random strength X_1 and is subjected to a random stress X_2. The component functions satisfactorily as long as $X_1 > X_2$, and fails when $X_1 < X_2$. The probability $R = Pr(X_1 > X_2)$ defines the component reliability. Stress-strength models have many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

Consider X_1 and X_2 to be independently distributed, with $X_1 \sim BTL(\gamma, \theta_1, \lambda_1, a_1, b_1)$ and $X_2 \sim BTL(\gamma, \theta_2, \lambda_2, a_2, b_2)$. The cdf $F_1$of X_1 and the pdf f_2 of X_2 are obtained from (10) and (11), respectively. Then,

$$R = Pr(X_1 > X_2) = \int_0^{\infty} f_2(y)[1 - F_1(y)]dy$$
$$= 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \int_0^{\infty} f_2(y)(1+\gamma y)^{-\theta(k+l)} dy$$
$$= \sum_{k,l=0}^{\infty} w_{kl}^{(1)}\, A(k,l),$$

where

$$w_{kl}^{(i)} = \sum_{j=0}^{\infty} (-1)^{j+k+l} \binom{b_i-1}{j} \binom{a_i+j}{k} \binom{a_i+j}{l} \frac{\lambda^l}{B(a,b)(a_i+j)}, \qquad i = 1,2,$$

and

$$A(k,l) = \int_0^{\infty} f_2(y)(1+\gamma y)^{-\theta(k+l)} dy.$$

Now,

$$A(k,l) = \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \int_0^{\infty} (\mathrm{r}+\mathrm{s})\,\gamma\,\theta_2 \left[(1+\gamma\, y)^{-[\theta_2(r+s)+\theta_2(k+l)]-1}\right] \mathrm{dy}$$
$$= \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{\gamma\,\theta_2(\mathrm{r}+\mathrm{s})}{\gamma\,\theta_1(\mathrm{k}+\mathrm{l}) + \gamma\,\theta_2(\mathrm{r}+\mathrm{s})}.$$

Hence,

$$R = 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{\gamma\,\theta_2(\mathrm{r}+\mathrm{s})}{\gamma\,\theta_1(\mathrm{k}+\mathrm{l}) + \gamma\,\theta_2(\mathrm{r}+\mathrm{s})}$$

$$= 1 + \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} w_k^{*(1)}\, w_r^{*(2)} \frac{r\gamma\, \theta_2}{k\gamma\, \theta_1 + r\gamma\, \theta_2}, \tag{26}$$

where

$$w_m^{*(i)} = \sum_{k,l:k+l=m} w_{kl}^{*(i)}, \quad i = 1,2.$$

5. Maximum likelihood estimation

Let $x_1, x_2, \dots, x_n$ be a random sample from the beta transmuted Lomax distribution with observed values $x_1, x_2, \dots, x_n$ and $\Theta = (\gamma, \theta, \lambda, a, b)$ be the parameter vector. The likelihood function $L(\Theta)$ is given by

$$L(\Theta) = \left(\frac{\gamma\theta}{B(a,b)}\right)^n \prod_{i=1}^{n} (1+\gamma x_i)^{-\theta-1} \left(1+\lambda-2\lambda(1+\gamma x_i)^{-\theta}\right)\left[\left(1-(1+\gamma x_i)^{-\theta}\right)\right]^{a-1}\left[\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)\right]^{a-1}\left[1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)\right]^{b-1}. \tag{27}$$

Then, the log-likelihood function $l(\Theta)$for the vector of parameters $\Theta = (\gamma, \theta, \lambda, a, b)$, is

$$l(\Theta) = n\log\gamma + n\log\theta - n\log[B(a,b)] + (-\theta-1)\sum_{i=1}^{n} log(1+\gamma x_i) + \sum_{i=1}^{n} \mathrm{Log}\left[1+\lambda-2\lambda(1+\gamma x_i)^{-\theta}\right] + (a-1)\left[\sum_{i=1}^{n} \mathrm{Log}\left(1-(1+\gamma x_i)^{-\theta}\right) + (a-1)\sum_{i=1}^{n} \mathrm{Log}\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)\right] + (b-1)\sum_{i=1}^{n} \mathrm{Log}\left[1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)\right]. \tag{28}$$

We differentiate (28) with respect to $\gamma, \theta, \lambda, a$ and b respectively to obtain the elements of score vector $\frac{\partial l(\Theta)}{\partial\Theta} = \left(\frac{\partial l(\Theta)}{\partial\gamma}, \frac{\partial l(\Theta)}{\partial\theta}, \frac{\partial l(\Theta)}{\partial\lambda}, \frac{\partial l(\Theta)}{\partial a}, \frac{\partial l(\Theta)}{\partial b}\right)^t$ as below

$$\frac{\partial l(\Theta)}{\partial\gamma} = \frac{n}{\gamma} - (\theta+1)\sum_{i=1}^{n} \frac{x_i}{1+\gamma x_i} + (a-1)\theta\sum_{i=1}^{n} \frac{x_i(1+\gamma x_i)^{-\theta-1}}{1-(1+\gamma x_i)^{-\theta}} + 2\theta\lambda\sum_{i=1}^{n} \frac{x_i(1+\gamma x_i)^{-\theta-1}}{(1+\lambda-2\lambda(1+\gamma x_i)^{-\theta})} - (a-1)\theta\lambda\sum_{i=1}^{n} \frac{x_i(1+\gamma x_i)^{-\theta-1}}{(1+\lambda(1+\gamma x_i)^{-\theta})} + (b-1)\theta\lambda\sum_{i=1}^{n} \frac{x_i(1+\gamma x_i)^{-\theta-1}\left(1-(1+\gamma x_i)^{-\theta}\right)}{1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)} - (b-1)\theta\sum_{i=1}^{n} \frac{x_i(1+\gamma x_i)^{-\theta-1}\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)}{1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)} \tag{29}$$

$$\frac{\partial l(\Theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \log(1+\gamma x_i) + (a-1)\sum_{i=1}^{n} \frac{(1+\gamma x_i)^{-\theta} log(1+\gamma x_i)}{1-(1+\gamma x_i)^{-\theta}} + 2\lambda\sum_{i=1}^{n} \frac{(1+\gamma x_i)^{-\theta} log(1+\gamma x_i)}{1+\lambda-2\lambda(1+\gamma x_i)^{-\theta}} - \lambda(a-1)\sum_{i=1}^{n} \frac{(1+\gamma x_i)^{-\theta} log(1+\gamma x_i)}{1+\lambda(1+\gamma x_i)^{-\theta}} + (b-1)\sum_{i=1}^{n} \frac{\lambda\left(1-(1+\gamma x_i)^{-\theta}\right)(1+\gamma x_i)^{-\theta} log(1+\gamma x_i)}{1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)} - (b-1)\sum_{i=1}^{n} \frac{\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)(1+\gamma x_i)^{-\theta} log(1+\gamma x_i)}{1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)} \tag{30}$$

$$\frac{\partial l(\Theta)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2(1+\gamma x_i)^{-\theta}}{1+\lambda-2\lambda(1+\gamma x_i)^{-\theta}} + (a-1)\sum_{i=1}^{n} \frac{(1+\gamma x_i)^{-\theta}}{1+\lambda(1+\gamma x_i)^{-\theta}} - (b-1)\sum_{i=1}^{n} \frac{\left(1-(1+\gamma x_i)^{-\theta}\right)(1+\gamma x_i)^{-\theta}}{1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)} \quad (31)$$

$$\frac{\partial l(\Theta)}{\partial a} = -n[\Psi(a) - \Psi(a+b)] + \sum_{i=1}^{n} \mathrm{Log}\left(1-(1+\gamma x_i)^{-\theta}\right) + \sum_{i=1}^{n} \mathrm{Log}\left(1+\lambda(1+\gamma x_i)^{-\theta}\right) \quad (32)$$

$$\frac{\partial l(\Theta)}{\partial b} = -n[\Psi(b) - \Psi(a+b)] + \sum_{i=1}^{n} \mathrm{Log}\left[1-\left(1-(1+\gamma x_i)^{-\theta}\right)\left(1+\lambda(1+\gamma x_i)^{-\theta}\right)\right], \quad (33)$$

where $\Psi(x)$ is the digamma function defined by $\Psi(x) = \frac{d \log \Gamma(x)}{dx}$, and $\Gamma(x)$ is the Gamma function.

For a random sample $(x_1, x_2, \dots, x_n)$ of size n from X, distributed with pdf (8), the sample log-likelihood is $l(\Theta) = \sum_{i=1}^{n} l_i(\Theta)$, where $l_i(\Theta)$ is the log-likelihood for the ith observation $(i = 1,2,\dots,n)$, and the score vector is

$$\frac{\partial l(\Theta)}{\partial \Theta} = \sum_{i=1}^{n} \frac{\partial l_i(\Theta)}{\partial \Theta}.$$

The maximum likelihood estimate (MLE) $\widehat{\Theta}$ of Θ is obtained by solving the system

$$\frac{\partial l(\Theta)}{\partial \Theta} = 0.$$

Under certain regularity conditions, $\sqrt{n}\left(\widehat{\Theta} - \Theta\right) \xrightarrow{d} N(0, I(\Theta)^{-1})$, (here $\xrightarrow{d}$ stands for convergence in distribution), where $I(\Theta)$ denotes the information matrix given by

$$I(\Theta) = E\left(\frac{\partial^2 l(\Theta)}{\partial \Theta \partial \Theta'}\right).$$

This information matrix $I(\Theta)$ may be approximated by the observed information matrix

$$I(\widehat{\Theta}) = E\left(\frac{\partial^2 l(\Theta)}{\partial \Theta \partial \Theta'}\right) |_{\Theta=\widehat{\Theta}}.$$

Then, using the approximation $\sqrt{n}\left(\widehat{\Theta} - \Theta\right) \sim N\left(0,\ I^{-1}\left(\widehat{\Theta}\right)\right)$, one can carry out tests and find confidence regions for functions of some or all parameters in Θ.

5.1. Simulation study

Here, we evaluate the performance of MLEs for the beta transmuted Lomax distribution. The assessments were based on simulation studies.

The assessment of the finite sample behaviour of MLEs for this distribution was based on the following:

1. use the inversion method to generate one thousand samples of size n from the BTL distribution, i.e. generate values of

$$X = \frac{\left[\frac{\lambda-1+\sqrt{(1+\lambda)^2-4\lambda\left(I_q^{-1}(a,b)\right)}}{2\lambda}\right]^{-\frac{1}{\theta}} - 1}{\gamma}$$

2. compute MLEs for one thousand samples, say $(\hat{\gamma}_i, \hat{\theta}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i)$ for $i = 1,2,\dots,1000$.

3. compute the standard errors of MLEs for one thousand samples, say $(S_{\hat{\gamma}_i}, S_{\hat{\theta}_i}, S_{\hat{\lambda}_i}, S_{\hat{a}_i}, S_{\hat{b}_i})$ for $i = 1,2,\dots,1000$. The standard errors were computed by inverting the observed information matrix.

4. compute the biases and mean squared errors by

$$Bias(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta)$$

$$MSE(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta)^2$$

For $\hat{\Theta}_i = (\hat{\gamma}_i, \hat{\theta}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i)$ and $\Theta = (\gamma, \theta, \lambda, a, b)$

5. We repeated these steps for $n = \{10,50,80,100,150,200,300\}$ with $\gamma = 0.9, \theta = 0.8, \lambda = 0.7, a = 0.6$ and $b = 0.5$, so computing $Bias(\hat{\Theta})$, $SE(\hat{\Theta})$ and $MSE(\hat{\Theta})$ for $\Theta = (\gamma, \theta, \lambda, a, b)$ and for $n = \{10,50,80,100,150,200,300\}$.

From Table 1 it is observed that as the sample size increases, the average biases, the standard error and the means squared errors decrease. This verifies the consistency properties of the estimates.

Table 1. Estimated parameters, Bias, standard error, and MSE of the BTL distribution

n	$\hat{\Theta}$	Bias	S.E	MSE
10	$\hat{\gamma} = 0.872032$	0.7720320	0.3391303	0.5960335
	$\hat{\theta} = 0.096215$	0.1037850	0.0089731	0.0107714
	$\hat{\lambda} = 0.721075$	0.4210746	1.9199212	0.1773038
	$\hat{a} = 0.725406$	0.3254056	0.0615352	0.1058888
	$\hat{b} = 0.670449$	0.1704485	0.0747316	0.0290527

Table 1. Estimated parameters, Bias, standard error, and MSE of the BTL distribution (cont.)

n	$\widehat{\Theta}$	Bias	S.E	MSE
50	$\hat{\gamma} = 0.861056$	0.7610562	0.0725915	0.5792066
	$\hat{\theta} = 0.153349$	0.0496510	0.0004640	0.0021763
	$\hat{\lambda} = 0.703408$	0.4034083	0.8194007	0.1627383
	$\hat{a} = 0.705059$	0.3050595	0.0126729	0.0940613
	$\hat{b} = 0.634201$	0.1342013	0.0124783	0.0180100
80	$\hat{\gamma} = 0.860920$	0.7609200	0.0482477	0.5789992
	$\hat{\theta} = 0.150779$	0.0494220	0.0002847	0.0021226
	$\hat{\lambda} = 0.702135$	0.4021352	0.7258839	0.1617127
	$\hat{a} = 0.705481$	0.3054812	0.0063866	0.0933187
	$\hat{b} = 0.634298$	0.1342981	0.0071282	0.0180036
100	$\hat{\gamma} = 0.860974$	0.7607430	0.0394231	0.5770819
	$\hat{\theta} = 0.151489$	0.0485110	0.0002302	0.0020533
	$\hat{\lambda} = 0.701706$	0.4017062	0.6649546	0.1613679
	$\hat{a} = 0.704974$	0.3049736	0.0045990	0.0930089
	$\hat{b} = 0.633523$	0.1335229	0.0054569	0.0178284
150	$\hat{\gamma} = 0.861044$	0.7604350	0.0273712	0.5761872
	$\hat{\theta} = 0.152401$	0.0475990	0.0001553	0.0016656
	$\hat{\lambda} = 0.701139$	0.4011389	0.5515348	0.1609124
	$\hat{a} = 0.704322$	0.3043215	0.0024106	0.0926116
	$\hat{b} = 0.632522$	0.1325220	0.0033301	0.0175621
200	$\hat{\gamma} = 0.861077$	0.7603690	0.0211253	0.5742380
	$\hat{\theta} = 0.152844$	0.0471560	0.0001171	0.0022237
	$\hat{\lambda} = 0.700853$	0.4008572	0.4719037	0.1606865
	$\hat{a} = 0.704005$	0.3040050	0.0014553	0.0924192
	$\hat{b} = 0.632034$	0.1320344	0.0023345	0.0174331
300	$\hat{\gamma} = 0.860907$	0.7601090	0.0146216	0.5712876
	$\hat{\theta} = 0.149021$	0.0467220	0.0000784	0.0021083
	$\hat{\lambda} = 0.708791$	0.4005767	0.3665787	0.1604617
	$\hat{a} = 0.709832$	0.3036949	0.0006557	0.0922306
	$\hat{b} = 0.641948$	0.1315549	0.0014122	0.0173067

6. Applications

In this section, we provide applications to three real data sets to illustrate the importance and potentiality of the BTL distribution and some of the models generated from Lomax distributions, namely the Lomax (L), transmuted Lomax (TL), beta Lomax (BL), Gamma Lomax (GaL), Marshall-Olkin Lomax (MOL) and Weibull Lomax (WL) distributions.

Data Set I: The first real data set (Ghitany et al. 2008) consists of 100 observations on waiting time (in minutes) before the customer received service in a bank. The data are: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 2.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5. The summary of the data set is provided as follows:

Table 2. Summary Statistics for Data Set I

Min.	Q_1	Median	Q_3	Mean	Max.	Var.	Skewness	Kurtosis
0.80	4.65	8.1	13.05	9.877	38.5	52.3741	1.47277	5.54029

Data Set II: The second data set (Gross and Clark (1975), page 105) on the relief times of twenty patients receiving an analgesic is: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. The summary of the data set is provided as follows:

Table 3. Summary Statistics for the Data Set II

Min.	Q_1	Median	Q_3	Mean	Max.	Var.	Skewness	Kurtosis.
1.1	1.45	1.7	2.1	1.9	38.5	0.4958	1.7198	5.9241

Data Set III : Here, we consider an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously studied by Lee and Wang (2003) and Lemonte and Cordeiro (2011). Bladder cancer is a disease in which abnormal cells multiply without control in the bladder. The most common type of bladder cancer recapitulates the normal histology of the urothelium and is known as transitional cell carcinoma. The data are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05. The summary statistics of this data set is given below:

Table 4. Summary Statistics for Data Set III

Min.	Q_1	Median	Q_3	Mean	Max.	Var.	Skewness	Kurtosis
0.08	3.335	6.395	11.885	9.366	79.05	110.425	3.287	18.483

From the descriptive statistics in Tables 2, 3 and 4 for the three data sets respectively, we observed that the three data sets are positively skewed, however, the third data set is highly peaked with a higher skewness coefficient followed by the second and then the first with a low peak. To compare this distribution, we have considered some criteria: the maximized log-likelihood (**−2*l***), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC). These statistics are given as:

$$AIC = 2K - 2l(\widehat{\Theta}), CAIC = AIC + \frac{2k(k+1)}{n-k-1}, BIC = klog(n) - 2l(\widehat{\Theta})$$

and

$$HQIC = 2klog(log(n)) - 2l(\widehat{\Theta}),$$

where k is the number of parameters in the statistical model, n the sample size and $l(\widehat{\Theta})$ is the log-likelihood function evaluated at the maximum likelihood estimates, Θ is the parameters. The distribution with minimum values for these statistics would be chosen as the best distribution to fit the data set in question.

Table 5. Criteria for comparison based on Data Set I

Models	**-2*l***	**AIC**	**CAIC**	**BIC**	**HQIC**	**Ranks**
L	818.3085	822.3085	822.4323	827.5189	824.4173	**7**
TL	732.5721	738.5721	738.8221	746.3876	741.7351	**4**
BL	735.5881	743.5881	744.0091	754.0088	747.8055	**5**
GaL	757.6042	767.6042	768.2425	780.63	772.8759	**6**
MOL	720.1842	730.1842	730.8225	743.2101	735.4560	**3**
WL	561.8530	571.8530	572.4914	584.8789	577.1249	**2**
BTL	386.4596	396.4596	397.0979	409.4854	401.731	**1**

Table 6. Criteria for comparison based on Data Set II

Models	**-2*l***	**AIC**	**CAIC**	**BIC**	**HQIC**	**Ranks**
L	264.3777	274.3777	278.6634	279.3563	275.3496	**7**
TL	250.2585	260.2585	264.5442	265.2372	261.2304	**6**
BL	243.1503	253.1503	257.4359	258.1289	254.1222	**4**
GaL	247.2609	257.2609	261.5466	262.2396	258.2328	**5**
MOL	234.0875	244.0875	248.3732	249.0663	245.0594	**3**
WL	229.1455	239.1455	243.4312	244.1241	240.1174	**2**
BTL	222.134	232.134	236.4197	237.1127	233.1059	**1**

Table 7. Criteria for comparison based on data set III

Models	**-2l**	**AIC**	**CAIC**	**BIC**	**HQIC**	**Ranks**
L	669.5493	679.5493	680.0411	693.8095	685.3433	**7**
TL	658.2596	668.2596	668.7514	682.5197	674.0535	**6**
BL	642.6644	652.6644	653.1562	666.9246	658.4584	**4**
GaL	651.7443	661.7443	662.2361	676.0045	667.5383	**5**
MOL	636.3210	646.3210	646.8128	660.5812	652.1149	**3**
WL	602.9675	612.9675	613.4593	627.2277	618.7615	**2**
BTL	566.9087	576.9087	577.4005	591.1689	582.7027	**1**

Tables 5, 6 and 7 provide corresponding values of the -2l**,** AIC, CAIC, BIC and HQIC for each of the distributions. The values of the statistics in all tables (5, 6 and 7) are lower for the BTL distribution followed by the WL and MOL distributions, which is an indication that the BTL distribution performed better than the other distributions considered in the analysis and could be chosen as the best model compared to the other distributions. This also provides additional evidence to the fact that generalizing probability distributions provides compound distributions that are more flexible compared to the parent distributions.

We have also considered a goodness-of-fit test in order to know which distribution has a better fit given some data sets. Hence, we apply the Anderson-Darling ($\boldsymbol{A}^*$), Cramrvon Mises ($\boldsymbol{W}^*$) and Kolmogrov-Smirnov (K-S) statistics. Further information about this statistics can be obtained from Al-Zahrani (2012). These statistics can be computed as:

$$A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[log\left(F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b})\right) + log\left(1 - F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b})\right)\right]^2$$

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b}) - \frac{2i-1}{2n}\right]^2$$

$$K-S = \max_i\left[\frac{i}{n} - F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b}), F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b}) - \frac{i-1}{n}\right]$$

where $F_{BTL}(x_i,\hat{\gamma},\hat{\theta},\hat{\lambda},\hat{a},\hat{b})$ is the empirical distribution function and n is the sample size. The distribution with minimum $\boldsymbol{A}^*$**,** $\boldsymbol{W}^*$ and K-S values is chosen as the best distribution to fit the data sets.

Table 8. Goodness-of-fit statistics based on data sets I, II and III.

Models	Data set I			Data set II			Data set III		
	A^*	W^*	K-S	A^*	W^*	K-S	A^*	W^*	K-S
L	1.7758	0.5738	0.4354	1.9524	1.542	0.0794	2.6287	0.6648	0.4381
TL	1.7386	0.4528	0.3957	1.7445	1.4726	0.0759	2.4113	0.4458	0.2175
BL	1.4738	0.3739	0.3018	0.3351	0.0846	0.0563	1.6615	0.1619	0.1277
GaL	1.5949	0.3974	0.3375	0.2754	0.0883	0.0636	1.6523	0.1558	0.1283
MOL	1.4185	0.2585	0.1774	0.2527	0.0857	0.0539	1.4086	0.1382	0.1216
WL	1.2528	0.2263	0,1517	0.2166	0.0808	0.0478	1.0637	0.1254	0.1137
BTL	1.1853	0.1347	0.1013	0.1873	0.0725	0.0473	0.8467	0.1198	0.1039

From the table above, we can observe the A^*, W^* and K-S values of the distributions based on data sets I, II and III. From the table, it is clear and we confirmed that BTL has smaller or lower values of the the A^*, W^* and K-S statistics for all the data sets compared to the WL, MOL, GaL, PL, TL and L distributions, which is an indication that it has a better performance compared to the other distributions. Hence, we can confidently conclude that the BTL distribution is better than the others.

7. Conclusions

In this study, we have introduced the so-called beta transmuted Lomax (BTL) distribution. This is a generalization of the transmuted Lomax distribution using the genesis of the beta distribution. Many distributions including Lomax, beta Lomax and transmuted Lomax are embedded in this newly developed BTL distribution. The mathematical properties of the new family including explicit expansions for the ordinary moments, quantiles, generating functions and order statistics have been provided. The model parameters have been estimated by the maximum likelihood estimation method and the observed information matrix has been determined. It has been shown, by means of a real data set, that special cases of the BTL distribution can provide better fits than other families of distributions.

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