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The Gamma Kumaraswamy-G family of distributions: theory, inference and applications

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ABSTRACT

In this paper, we introduce a new family of univariate continuous distributions called the Gamma Kumaraswamy-generated family of distributions. Most of its properties are studied in detail, including skewness, kurtosis, analytical comportments of the main functions, moments, stochastic ordering and order statistics. The next part of the paper focuses on a particular member of the family with four parameters, called the gamma Kumaraswamy exponential distribution. Among its advantages, the following should be mentioned: the corresponding probability density function can have symmetrical, left-skewed, right-skewed and reversed-J shapes, while the corresponding hazard rate function can have (nearly) constant, increasing, decreasing, upside-down bathtub, and bathtub shapes. Subsequently, the inference on the gamma Kumaraswamy exponential model is performed. The method of maximum likelihood is applied to estimate the model parameters. In order to demonstrate the importance of the new model, analyses on two practical data sets were carried out. The results proved more favourable for the studied model than for any of the other eight competitive models.

Key words: Kumaraswamy distribution, gamma distribution, generalised family, moments, stochastic ordering, maximum likelihood method, data analysis.

1. Introduction

In order to meet scientific requirements, modern experiments require high precision in data analysis. Unfortunately, in most situations this requirement cannot be achieved through the use of standard statistical models. For this reason, the creation of new flexible models, well adapted to the context, remains a passionate challenge for the statisticians. From a probabilistic point of view, attractive models can be derived from families of distributions enjoying desirable properties. Such families can be defined by the use of effective techniques introducing tuning parameters to well-established distributions. These families are often characterized by sophisticated but flexible functions, which can be handled thanks to

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the computational and analytical facilities available in modern programming software (as R, Maple, Mathematica...). In particular, the use of this software can easily tackle the problems involved in computing eventual special functions. Among the high impacted families of distributions, there are the beta-G family by Eugene *et al.* (2002) and Jones (2004), the Kumaraswamy-G (Kw-G) family by Cordeiro and de Castro (2011) and Ramos (2014), the Kumaraswamy Poisson-G (Kw-G) family by Ramos (2014), the McDonald-G (Mc-G) family by Alexander *et al.* (2012), the gamma-G type 1 family by Zografos and Balakrishnan (2009) and Amini *et al.* (2014), the gamma-G type 2 family by Ristic and Balakrishnan (2012) and Amini *et al.* (2014), the odd-gamma-G type 3 family by Torabi and Montazari (2012), the logistic-G family by Torabi and Montazari (2014), the odd exponentiated generated (odd exp-G) family by Cordeiro *et al.* (2013), the transformed-transformer (T-X) (Weibull-X and gamma-X) family by Alzaatreh *et al.* (2013a), the exponentiated T-X family by Alzaatreh *et al.* (2013b), the odd Weibull-G family by Bourguignon *et al.* (2014), the exponentiated half-logistic by Cordeiro *et al.* (2014), the T-X{Y}-quantile based approach family by Aljarrah *et al.* (2014), the T-R{Y} family by Alzaatreh *et al.* (2014), the odd Burr-III-G family by Jamal *et al.* (2017), the Kumaraswamy odd Burr-G family by Nasir *et al.* (2018), the generalized odd gamma-G family by Hosseini *et al.* (2018), the truncated Cauchy power-G family by Aldahlan *et al.* (2019) and the type II general inverse exponential-G family by Jamal *et al.* (2020).

In this study, we introduce a new family of distributions derived to two important families: the Kumaraswamy-G and odd gamma-G families introduced by Cordeiro and de Castro (2011) and Torabi and Montazari (2012), respectively. Before going further in the motivation, let us briefly describe these two well-recognized families, beginning with the Kumaraswamy-G family of distributions. Let $a > 0$, $b > 0$, $G(x)$ be the cumulative distribution function (cdf) of an univariate continuous distribution and $g(x)$ be the corresponding probability distribution function (pdf). Then, the Kumaraswamy-G family of distributions is characterized by the cdf given by

$$H(x) = 1 - \{1 - G(x)^a\}^b, \qquad x \in \mathbb{R} \tag{1}$$

and the corresponding pdf can be expressed as

$$h(x) = abg(x)G(x)^{a-1}\{1 - G(x)^a\}^{b-1}, \qquad x \in \mathbb{R}. \tag{2}$$

Thus, the feature of the Kumaraswamy-G family is to add two shape parameters to the former distribution characterized by the cdf $G(x)$, increasing mechanically its flexible properties. This allows the construction of more flexible models to analyse a wide variety of data sets, as developed in Cordeiro and de Castro (2011) for the normal, Weibull, gamma, Gumbel and inverse Gaussian distributions. The Kumaraswamy-G family of distributions is also known to be a simple alternative to the beta-G family of distribution established by Eugene *et al.* (2002). The essentials of the standard Kumaraswamy distribution are detailed in Jones (2008). Current developments and extensions of the Kumaraswamy-G family of distributions can be found in, e.g. Paranaiba *et al.* (2012), de Pascoa *et al.* (2011), Ramos (2014), Gomes *et al.* (2014), Rodrigues and Silva (2015) and Jamal *et al.* (2019).

On the other side, Torabi and Montazari (2012) introduced the odd gamma-G family of distributions, briefly described below. Let $\alpha > 0$, $H(x)$ be the cdf of an univariate continuous distribution, $\bar{H}(x) = 1 - H(x)$ and $h(x)$ be the corresponding pdf. Let $\gamma_1(\alpha, z)$ be the regularized lower incomplete gamma function defined by $\gamma_1(\alpha, z) = \gamma(\alpha, z)/\Gamma(\alpha)$, where $\gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt$ and $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$. Then, the odd gamma-G family of distributions ”with $G = H$” is characterized by the cdf given as

$$F(x) = \gamma_1 \left(\alpha, \frac{H(x)}{\bar{H}(x)} \right), \qquad x \in \mathbb{R} \tag{3}$$

and the corresponding pdf is specified by

$$f(x) = \frac{1}{\Gamma(\alpha)} \frac{h(x) H(x)^{\alpha-1}}{\bar{H}(x)^{\alpha+1}} \exp\left(-\frac{H(x)}{\bar{H}(x)} \right), \qquad x \in \mathbb{R}. \tag{4}$$

The odd-gamma-G family of distributions gives an alternative to the useful gamma-G type 1 family of distributions introduced by Zografos and Balakrishnan (2009) in the following stochastic ordering sense: $F(x) \geq K(x)$, where $K(x) = \gamma_1(\alpha, -\log[\bar{H}(x)])$ is the cdf corresponding to the gamma-G type 1 family of distributions. Also, the merits of the odd-gamma-G family have been highlighted in recent studies, including those of Torabi and Montazari (2012), Hosseini *et al.* (2018), Oluyede *et al.* (2018) and Nasir *et al.* (2020), via the exploration of various theoretical and practical aspects. In particular, it is shown that the parental distribution characterized by the cdf $H(x)$ can take the benefits of the considered polynomial-exponential transformation with α as the tuning parameter, allowing the construction of new flexible statistical models. In particular, for appropriated $H(x)$, the analyses of a wide broad range of real life data sets are favourable to the odd-gamma-G models in comparison to well-recognized competitors.

In the light of the previous arguments, a promising direction of work becomes the combination of the Kumaraswamy-G and odd gamma-G families via the composition technique of the respective cdfs. Thus, we aim to create a new generalized family of distributions benefiting of the respective qualities of these two families, aiming

- to skew any symmetrical distribution;
- to modulate the weight of the tails of any parental distribution;
- to increase the possible shapes of the (probabilistic or reliability) functions of the parental distribution;
- to construct new statistical models with better (fits) properties than other competitive models, or enlarging the horizon of fields of applications.

The proposed family is called the gamma Kumaraswamy-G (GKw-G) family of distributions. This study explores, in both theoretical and practical terms, the properties of the GKw-G family. A special member defined with the exponential distribution as the parent, called the GKw-E distribution, will serve as a statistical model. The complete analyses of two practical data sets are proposed, showing that the GKw-E model presents better fit to eight notorious models in the field.

The rest of the article is organized as follows. In Section 2, we present the main functions and properties of the GKw-G family of distributions. In Section 3, the GKw-E distribution is introduced, as well as some of its structural properties. In Section 4, the GKw-E model parameters are estimated by the maximum likelihood method and a simulation study is performed to verify the convergence properties. Also, the usefulness of the GKw-E model is illustrated by means of two practical data sets. Finally, Section 5 offers some concluding remarks.

2. The gamma Kumaraswamy-G family of distributions

2.1. Presentation

We characterize the GKw-G family of distributions by the cdf of the odd gamma-H family of distributions given by (3), defined with the cdf $H(x)$ of the Kumaraswamy-G family of distributions given as (1). Hence, by noticing that $H(x)/\bar{H}(x) = \{1-G(x)^a\}^{-b} - 1$, the corresponding cdf is defined by

$$F(x) = \gamma_1\left(\alpha, \{1-G(x)^a\}^{-b} - 1\right), \qquad x \in \mathbb{R}. \tag{5}$$

One can remark that, if $b = 1$, this cdf becomes the one of the generalized odd gamma-G family introduced by Hosseini *et al.* (2018), that is $F(x) = \gamma_1(\alpha, G(x)^a/[1-G(x)^a])$, $x \in \mathbb{R}$. In this sense, the GKw-G family of distributions can be viewed as a generalization of this family. The parameter b plays an important role, as we shall see later. The corresponding survival (sf) function is

$$S(x) = 1 - \gamma_1\left(\alpha, \{1-G(x)^a\}^{-b} - 1\right), \qquad x \in \mathbb{R}.$$

The pdf of the GKw-G family can be obtained by putting (1) and (2) into (4). More directly, upon almost everywhere differentiation of $F(x)$, it is obtained as

$$\begin{aligned} f(x) = {} & \frac{ab}{\Gamma(\alpha)} g(x) G(x)^{a-1} \{1-G(x)^a\}^{-b-1} \left\{\{1-G(x)^a\}^{-b} - 1\right\}^{\alpha-1} \\ & \times \exp\left[1 - \{1-G(x)^a\}^{-b}\right]. \qquad x \in \mathbb{R}. \end{aligned} \tag{6}$$

The corresponding hazard rate function (hrf) is obtained as $\pi(x) = f(x)/S(x)$, that is

$$\pi(x) = \frac{ab}{\Gamma(\alpha)} \frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}\left\{\{1-G(x)^a\}^{-b}-1\right\}^{\alpha-1}\exp\left[1-\{1-G(x)^a\}^{-b}\right]}{1-\gamma_1\left(\alpha, \{1-G(x)^a\}^{-b}-1\right)}.$$

Some special members of the GKw-G family characterized by their cdfs are presented in Table 1.

Table 1: Some members of the GKw-G family of distributions characterized by their cdfs.

cdf $G(x)$	Support	GKw-G cdf $F(x)$	Parameters
Uniform	$(0,\theta)$	$\gamma_1\left(\alpha,\{1-(x/\theta)^a\}^{-b}-1\right)$	(α,a,b,θ)
Exponential	$(0,+\infty)$	$\gamma_1\left(\alpha,\left\{1-[1-e^{-\lambda x}]^a\right\}^{-b}-1\right)$	(α,a,b,λ)
Weibull	$(0,+\infty)$	$\gamma_1\left(\alpha,\left\{1-[1-e^{-(\lambda x)^\beta}]^a\right\}^{-b}-1\right)$	(α,a,b,λ)
Inverse Weibull	$(0,+\infty)$	$\gamma_1\left(\alpha,\left\{1-e^{-a(\lambda/x)^\beta}\right\}^{-b}-1\right)$	$(\alpha,a,b,\lambda,\beta)$
Burr XII	$(0,+\infty)$	$\gamma_1\left(\alpha,\{1-\{1-[1+(x/s)^c]^{-k}\}^a\}^{-b}-1\right)$	(α,a,b,c,k,s)
Logistic	$\mathbb{R}$	$\gamma_1\left(\alpha,\left\{1-[1+e^{-(x-\mu)/s}]^{-a}\right\}^{-b}-1\right)$	(α,a,b,μ,s)
Gumbel	$\mathbb{R}$	$\gamma_1\left(\alpha,\left\{1-\exp(-ae^{-(x-\mu)/\sigma})\right\}^{-b}-1\right)$	(α,a,b,μ,σ)
Normal	$\mathbb{R}$	$\gamma_1\left(\alpha,\{1-\Phi((x-\mu)/\sigma)^a\}^{-b}-1\right)$	(α,a,b,μ,σ)
Cauchy	$\mathbb{R}$	$\gamma_1\left(\alpha,\{1-[(1/\pi)\arctan((x-x_0)/\theta)+1/2]^a\}^{-b}-1\right)$	(α,a,b,x_0,θ)

Thanks to its simplicity in the definition, the special member of the GKw-G family based on the exponential distribution will be the object of all the attention in our applications.

Let $Q_G(x)$ be the quantile function corresponding to $G(x)$, that is, the function satisfying the following equation: $G(Q_G(p)) = Q_G(G(p)) = p$ for any $p \in (0,1)$. Then, the quantile function of the GKw-G family of distributions can be expressed as

$$Q(p) = Q_G\left(\left[1-\left\{1+\gamma_1^{-1}(\alpha,p)\right\}^{-1/b}\right]^{1/a}\right), \qquad p \in (0,1), \tag{7}$$

where $\gamma_1^{-1}(\alpha,p)$ denotes the inverse function of $\gamma_1(\alpha,p)$, i.e., satisfying $\gamma_1\left(\alpha,\gamma_1^{-1}(\alpha,p)\right) = \gamma_1^{-1}(\alpha,\gamma_1(\alpha,p)) = p$ for any $p \in (0,1)$. Further details on $\gamma_1^{-1}(\alpha,p)$ can be found in (Abramowitz and Stegun, 1965, Section 6.5). In particular, the median of the GKw-G family is specified by $M = Q(1/2)$. Also, the three quartiles are defined by $Q_1 = Q(1/4)$, $Q_2 = M$ and $Q_3 = Q(3/4)$, and the seven octiles by $O_1 = Q(1/8)$, $O_2 = Q(2/8) = Q_1$, $O_3 = Q(3/8)$, $O_4 = Q(4/8)$, $O_5 = Q(5/8)$, $O_6 = Q(6/8) = Q_3$ and $O_7 = Q(7/8)$.

The quantile function and its related values are useful to evaluate some properties of the GKw-G family, such as the skewness and kurtosis, as described below.

2.2. Skewness and kurtosis

A measure of the skewness of the GKw-G family is given by

$$S = \frac{Q_3+Q_1-2Q_2}{Q_3-Q_1}. \tag{8}$$

In full generality, for given $G(x)$, α, a and b, when the corresponding GKw-G distribution is symmetric, we have $S = 0$, when it is right skewed, we have $S > 0$ and when it is left skewed, we have $S < 0$. See Kenney and Keeping (1962).

Also, a measure of the kurtosis of the GKw-G family of distributions is proposed by

$$K = \frac{O_3 - O_1 + O_7 - O_5}{O_6 - O_2}. \tag{9}$$

For given $G(x)$, α, a and b, as K increases, the tail of the corresponding GKw-G distribution becomes heavier. We refer to Moors (1998).

The advantages of these measures are to be robust in presence of outliers and they always exist (even if the distribution does not admit moments).

2.3. Properties

Diverse and important properties of the new family are now described.

2.3.1 Asymptotic properties

The two following propositions investigate the asymptotic properties of the cdf, sf, pdf and hrf of the GKw-G family of distributions.

Proposition 2.1 *The asymptotic equivalences of the cdf, pdf and hrf of the GKw-G family when $G(x) \to 0$ are, respectively,*

$$F(x) \sim \frac{b^{\alpha}}{\alpha\Gamma(\alpha)} G(x)^{a\alpha}, \qquad f(x) \sim \frac{ab^{\alpha}}{\Gamma(\alpha)} g(x) G(x)^{a\alpha-1}, \qquad h(x) \sim \frac{ab^{\alpha}}{\Gamma(\alpha)} g(x) G(x)^{a\alpha-1}.$$

Proof 2.1 *The proof follows from the following equivalences: when $y \to 0$, we have $(1-y^a)^{-b} \sim 1 + by^a$ and $\gamma_1(\alpha, y) \sim y^{\alpha}/(\alpha\Gamma(\alpha))$.*

Proposition 2.2 *The asymptotic equivalences of the sf, pdf and hrf of the GKw-G family when $G(x) \to 1$ are, respectively,*

$$S(x) \sim \frac{a^{-b(\alpha-1)}}{\Gamma(\alpha)} \{1 - G(x)\}^{-b(\alpha-1)} e^{1-a^{-b}\{1-G(x)\}^{-b}},$$

$$f(x) \sim \frac{ba^{-\alpha b}}{\Gamma(\alpha)} g(x) \{1 - G(x)\}^{-\alpha b - 1} e^{1-a^{-b}\{1-G(x)\}^{-b}}$$

and

$$h(x) \sim ba^{-b} g(x) \{1 - G(x)\}^{-b-1}.$$

Proof 2.2 *The proof follows from the following equivalences: when $y \to +\infty$, we have $\gamma_1(\alpha, y) \sim 1 - y^{\alpha-1}e^{-y}/\Gamma(\alpha)$ and, when $y \to 1$, we have $y^a \sim 1 - a(1-y)$.*

Propositions 2.1 and 2.2 are useful to understand the roles of $G(x)$, $g(x)$, α, a and b on the asymptotic properties of the cdf, sf, pdf and hrf of the GKw-G family. In particular, we see that b has a strong impact, mainly when $G(x) \to 1$.

2.3.2 Critical points

The analytical study of the pdf and hrf of the GKw-G family is crucial to understand their complexity. The critical points are essential in this regard. As usual, they can be determined by solving the following nonlinear equations $\partial \log[f(x)]/\partial x = 0$ and $\partial \log[h(x)]/\partial x = 0$, respectively, both obtained as

$$\frac{\partial g(x)/\partial x}{g(x)} + (a-1)\frac{g(x)}{G(x)} + a(b+1)\frac{g(x)G(x)^{a-1}}{1-G(x)^a} + ab(\alpha-1)\frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}}{\{1-G(x)^a\}^{-b}-1} - abg(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} = 0 \quad (10)$$

and

$$\frac{\partial g(x)/\partial x}{g(x)} + (a-1)\frac{g(x)}{G(x)} + a(b+1)\frac{g(x)G(x)^{a-1}}{1-G(x)^a} + ab(\alpha-1)\frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}}{\{1-G(x)^a\}^{-b}-1} - abg(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} + \frac{ab}{\Gamma(\alpha)}\frac{g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}\left\{\{1-G(x)^a\}^{-b}-1\right\}^{\alpha-1}\exp\left[1-\{1-G(x)^a\}^{-b}\right]}{1-\gamma_1\left(\alpha,\{1-G(x)^a\}^{-b}-1\right)} = 0. \quad (11)$$

The nature of the obtained critical points can be determined by investigating the signs of $\partial^2 \log[f(x)]/\partial x^2$ and $\partial^2 \log[h(x)]/\partial x^2$ taken at these points, respectively.

2.3.3 Some results in distribution

As usual, for any random variable U following the uniform distribution over $(0,1)$, the random variable X defined by $X = Q(U)$ has the cdf $F(x)$. For given $G(x)$, α, a and b, this characterization is useful to generate random values distributed according to the related GKw-G distribution through the inverse transform sampling.

Now, we say that a random variable follows the gamma distribution $\mathcal{G}_{am}(1,\alpha)$ if it has the cdf given by $K(x) = \gamma_1(\alpha,x)$, $x > 0$. If X is a random variable having the cdf of the GKw-G family, then the random variable Y defined by $Y = \{1-G(X)^a\}^{-b} - 1$ follows the gamma distribution $\mathcal{G}_{am}(1,\alpha)$.

Also, if Y is a random variable following the gamma distribution $\mathcal{G}_{am}(1,\alpha)$, then the random variable X defined by $X = Q_G\left(\left[1-\{1+Y\}^{-1/b}\right]^{1/a}\right)$ has the cdf of the GKw-G family.

2.3.4 Linear representations

This subsection is devoted to exploitable linear representations for the cdf and pdf of the GKw-G family.

Proposition 2.3 *We have the following linear representations for the cdf and pdf of the GKw-G family of distributions:*

$$F(x) = \sum_{i=0}^{+\infty} w_i G(x)^{ai}, \quad f(x) = \sum_{i=1}^{+\infty} w_i \left[aig(x)G(x)^{ai-1} \right], \tag{12}$$

where

$$w_i = \sum_{j,k=0}^{+\infty} \frac{(-1)^{i+j+k}}{\Gamma(\alpha)k!(\alpha+k)} \binom{\alpha+k}{j} \binom{b(j-\alpha-k)}{i}$$

and $\binom{b}{a}$ *denotes the generalized binomial coefficient, i.e.* $\binom{b}{a} = b(b-1)\ldots(b-a+1)/a!$.

Proof 2.3 *By using the regularized lower incomplete gamma function series expansion, i.e.*

$$\gamma_1(\alpha, y) = \sum_{k=0}^{+\infty} (-1)^k \frac{y^{\alpha+k}}{\Gamma(\alpha)k!(\alpha+k)}, \qquad y \geq 0,$$

and after some simplifications, we can express $F(x)$ *as*

$$\begin{aligned} F(x) &= \gamma_1 \left(\alpha, \frac{1-\{1-G(x)^a\}^b}{\{1-G(x)^a\}^b} \right) \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{\Gamma(\alpha)k!(\alpha+k)} \{1-G(x)^a\}^{-b(\alpha+k)} \underbrace{\left[1-\{1-G(x)^a\}^b\right]^{\alpha+k}}_{A}. \end{aligned}$$

By virtue of the generalized binomial series expansion, the term A can expressed as

$$A = \sum_{j=0}^{+\infty} (-1)^j \binom{\alpha+k}{j} \{1-G(x)^a\}^{bj}.$$

By putting the previous equalities together, we get

$$F(x) = \sum_{j,k=0}^{+\infty} \frac{(-1)^{j+k}}{\Gamma(\alpha)k!(\alpha+k)} \binom{\alpha+k}{j} \underbrace{\{1-G(x)^a\}^{b(j-\alpha-k)}}_{B}.$$

By using again the generalized binomial series expansion, we get

$$B = \sum_{i=0}^{+\infty} (-1)^i \binom{b(j-\alpha-k)}{i} G(x)^{ai}.$$

The desired linear representation of $F(x)$ *follows from the combination of all the equalities above. Upon differentiation, we derive the linear representation of* $f(x)$*. This completes the proof of Proposition 2.3.*

Since it depends on the well-known exp-G family of distributions (with parameter *ai* for any integer i), the linear representations presented in Proposition 2.3 are useful to derive related analytical and numerical properties. Some of them are explored in the subsections below.

2.3.5 Moments and derivations

Here, we assume that all the presented integrals and sum exist (which is not necessarily the case, depending on the definition of $G(x)$, among others). Let r be an integer. Then, the r-th ordinary moment of the GKw-G family is given as

$$\mu_r' = \int_{-\infty}^{+\infty} x^r f(x)dx = \int_{-\infty}^{+\infty} x^r \frac{ab}{\Gamma(\alpha)} g(x)G(x)^{a-1}\left\{1-G(x)^a\right\}^{-b-1}\left\{\left\{1-G(x)^a\right\}^{-b}-1\right\}^{\alpha-1}$$
$$\times \exp\left[1-\left\{1-G(x)^a\right\}^{-b}\right] dx.$$

By using the quantile function in (7), with the change of variable $x = Q(p)$, we can express μ_r' as

$$\mu_r' = \int_0^1 Q(p)^r dp = \int_0^1 \left[Q_G\left(\left[1-\left\{1+\gamma_1^{-1}(\alpha,p)\right\}^{-1/b}\right]^{1/a}\right)\right]^r dp.$$

For given $G(x)$, r, α, a and b, this integral can be computed numerically via any mathematical software (R, Maple, Matlab, Mathematica...). Also, a linear representation of μ_r' can be deduced from Proposition 2.3. Indeed, owing to (12), we have

$$\mu_r' = \sum_{i=1}^{+\infty} w_i \int_{-\infty}^{+\infty} x^r \left[aig(x)G(x)^{ai-1}\right] dx = \sum_{i=1}^{+\infty} w_i ai \int_0^1 p^{ai-1} Q_G(p)^r dp.$$

Among others, one can deduce the mean defined by $\mu = \mu_1'$, the variance given by $\sigma^2 = \mu_2' - (\mu_1')^2$, the r-th central moment given as

$$\mu_r = \int_{-\infty}^{+\infty} (x-\mu_1')^r f(x)dx = \sum_{k=0}^{r} \binom{r}{k} (-1)^k (\mu_1')^k \mu_{r-k}', \tag{13}$$

the coefficient of skewness given as $CS = \mu_3/\mu_2^{3/2}$, the coefficient of kurtosis obtained as $CK = \mu_4/\mu_2^2$ and the moment generating function given by

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx = \sum_{r=0}^{+\infty} \frac{t^r}{r!}\mu_r'.$$

Alternatively, we can use (12) to have a linear representation for $M(t)$ without using moments. Indeed, we have

$$M(t) = \sum_{i=1}^{+\infty} w_i \int_{-\infty}^{+\infty} e^{tx} \left[aig(x)G(x)^{ai-1}\right] dx = \sum_{i=1}^{+\infty} w_i ai \int_0^1 p^{ai-1} e^{tQ_G(p)} dp.$$

Finally, let us mention that the incomplete moments can be expressed in a similar way, giving expressions for the Bonferroni and Lorenz curves, mean residual-life, mean waiting-time, mean deviation about the mean and mean deviation about the median. For similar developments, we refer to the methodology of Hosseini *et al.* (2018).

2.3.6 Stochastic ordering

We now prove a result on the stochastic ordering involving the GKw-G family of distributions with a and b as common parameters. Further details on stochastic ordering can be found in Shaked and Shanthikumar (1994).

Proposition 2.4 *Let X be a random variable having the pdf $f_1(x)$ given by* (6) *with parameters α_1, a and b and Y be a random variable having the pdf $f_2(x)$ given by* (6) *with parameters α_2, a and b. Then, if $\alpha_1 \leq \alpha_2$, we have $X \leq_{lr} Y$, i.e. $f_1(x)/f_2(x)$ is decreasing.*

Proof 2.4 *We have*

$$\frac{f_1(x)}{f_2(x)} = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)}\left\{\{1-G(x)^a\}^{-b}-1\right\}^{\alpha_1-\alpha_2}.$$

By differentiating with respect to x, since $\alpha_1 \leq \alpha_2$, we have

$$\frac{\partial}{\partial x}\frac{f_1(x)}{f_2(x)} = \\ \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)}(\alpha_1-\alpha_2)\left\{\{1-G(x)^a\}^{-b}-1\right\}^{\alpha_1-\alpha_2-1} abg(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1} \leq 0.$$

Hence, we have $X \leq_{lr} Y$. This ends the proof of Proposition 2.4.

2.4. Order statistics

The order statistics naturally arise in many applications involving data relating to survival testing studies. All the details can be found in the book of David and Nagaraja (2003). This subsection is devoted to the order statistics of the GKw-G family. Let $X_1,\ldots,X_n$ be the random sample from the GKw-G family and $X_{i:n}$ be the i-th order statistic. Then, the pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i}, \qquad x \in \mathbb{R}. \tag{14}$$

Hence, by using (5) and (6), we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\frac{ab}{\Gamma(\alpha)}g(x)G(x)^{a-1}\{1-G(x)^a\}^{-b-1}\left\{\{1-G(x)^a\}^{-b}-1\right\}^{\alpha-1} \\ \exp\left[1-\{1-G(x)^a\}^{-b}\right]\gamma_1\left(\alpha,\{1-G(x)^a\}^{-b}-1\right)^{i-1}\left[1-\gamma_1\left(\alpha,\{1-G(x)^a\}^{-b}-1\right)\right]^{n-i}.$$

In particular, the pdfs of $X_{1:n} = \inf(X_1,\ldots,X_n)$ and $X_{n:n} = \sup(X_1,\ldots,X_n)$ are given by $f_{1:n}(x)$ and $f_{n:n}(x)$, respectively.

The proposition below presents a result characterizing $f_{i:n}(x)$.

Proposition 2.5 *The pdf of $X_{i:n}$ can be expressed as a linear combination of pdfs of the exp-G family of distributions.*

Proof 2.5 *Let us consider the expression of $f_{i:n}(x)$ given by* (14). *It follows from the binomial formula and* (12) *that*

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j f(x)F(x)^{j+i-1}$$
$$= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left\{\sum_{\ell=1}^{+\infty} w_\ell\left[a\ell g(x)G(x)^{a\ell-1}\right]\right\}\left[\sum_{k=0}^{+\infty} w_k G(x)^{ak}\right]^{j+i-1}.$$

By virtue of a result established by (Gradshteyn and Ryzhik, 2000, Section 0.314), we have

$$\left[\sum_{k=0}^{+\infty} w_k G(x)^{ak}\right]^{j+i-1} = \sum_{m=0}^{+\infty} d_{j+i-1,m}G(x)^{am},$$

where $d_{j+i-1,0} = w_0^{j+i-1}$ and, for any integer $m \geq 1$,

$$d_{j+i-1,m} = \frac{1}{mw_0}\sum_{k=1}^{m}(k(j+i)-m)w_k d_{j+i-1,m-k}.$$

By putting the equalities above together, we obtain

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{\ell=1}^{+\infty}\sum_{m=0}^{+\infty}\binom{n-i}{j}(-1)^j w_\ell d_{j+i-1,m}\frac{\ell}{\ell+m}q_{\ell,m}(x), \qquad (15)$$

where $q_{\ell,m}(x) = a(\ell+m)g(x)G(x)^{a(\ell+m)-1}$. Since $q_{\ell,m}(x)$ is a pdf of the exp-G family with parameter $a(\ell+m)$, the proof of Proposition 2.5 is complete.

By using the existing results on the exp-G family, we can use Proposition 2.5 to derive mathematical properties of the distribution of the *i*-th order statistics, as moments and all the related quantities.

3. GKw-Exponential distribution

3.1. Definition

In this section, we focus our attention on the special member of the GKw-G family based on the exponential distribution. Hence, by substituting the cdf $G(x) = 1-e^{-\lambda x}$, $x > 0$, into (5), the cdf of this special distribution is given by

$$F_{GKw-E}(x) = \gamma_1\left(\alpha, \left\{1-\left(1-e^{-\lambda x}\right)^a\right\}^{-b}-1\right), \quad x > 0. \qquad (16)$$

The related distribution is called the GKw-Exponential (GKw-E) distribution. Naturally, the corresponding sf is

$$S_{GKw-E}(x) = 1 - \gamma_1\left(\alpha, \left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b} - 1\right), \quad x > 0.$$

The corresponding pdf is specified by

$$f_{GKw-E}(x) = \frac{ab\lambda}{\Gamma(\alpha)} e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}\left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b-1}\left\{\left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b} - 1\right\}^{\alpha-1} \times \exp\left[1 - \left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b}\right]. \quad x > 0, \tag{17}$$

and the corresponding hrf is given as

$$\pi_{GKw-E}(x) = \frac{ab\lambda}{\Gamma(\alpha)} \frac{e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{a-1}\left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b-1}\left\{\left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b} - 1\right\}^{\alpha-1}}{1 - \gamma_1\left(\alpha, \left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b} - 1\right)} \times \exp\left[1 - \left\{1 - \left(1 - e^{-\lambda x}\right)^a\right\}^{-b}\right], \quad x > 0. \tag{18}$$

Let us now investigate some asymptotic properties of $F_{GKw-E}(x)$, $S_{GKw-E}(x)$, $f_{GKw-E}(x)$ and $h_{GKw-E}(x)$. When $x \to 0$, we have

$$F_{GKw-E}(x) \sim \frac{b^\alpha \lambda^{a\alpha}}{\alpha\Gamma(\alpha)} x^{a\alpha}, \quad f_{GKw-E}(x) \sim \frac{ab^\alpha \lambda^{a\alpha}}{\Gamma(\alpha)} x^{a\alpha-1}, \quad h_{GKw-E}(x) \sim \frac{ab^\alpha \lambda^{a\alpha}}{\Gamma(\alpha)} x^{a\alpha-1}.$$

The following limits follow. If $a\alpha < 1$, we have $f_{GKw-E}(x) \to +\infty$, if $a\alpha = 1$, we have $f_{GKw-E}(x) \to ab^{1/a}\lambda/\Gamma(\alpha)$, and if $a\alpha > 1$, we have $f_{GKw-E}(x) \to 0$. Similarly, if $a\alpha < 1$, we have $h_{GKw-E}(x) \to +\infty$, if $a\alpha = 1$, we have $h_{GKw-E}(x) \to ab^{1/a}\lambda/\Gamma(\alpha)$, and if $a\alpha > 1$, we have $h_{GKw-E}(x) \to 0$. When $x \to +\infty$, we have

$$S_{GKw-E}(x) \sim \frac{a^{-b(\alpha-1)}}{\Gamma(\alpha)} e^{\lambda b(\alpha-1)x} e^{1-a^{-b}e^{\lambda bx}}, \qquad f_{GKw-E}(x) \sim \frac{\lambda b a^{-\alpha b}}{\Gamma(\alpha)} e^{\lambda b\alpha x} e^{1-a^{-b}e^{\lambda bx}}$$

and

$$h_{GKw-E}(x) \sim \lambda b a^{-b} e^{\lambda bx}.$$

Hence, we have $f_{GKw-E}(x) \to 0$ and $h_{GKw-E}(x) \to +\infty$.

In order to give more concrete illustrations on their shapes, Figure 1 displays some plots of the GKw-E pdf and hrf for specified parameters values. It indicates that the GKw-E distribution can be right-skewed, left-skewed and reversed-J shaped, whereas the GKw-E hrf can produce various shapes such as increasing, decreasing, bathtub and upside-down bathtub shapes.

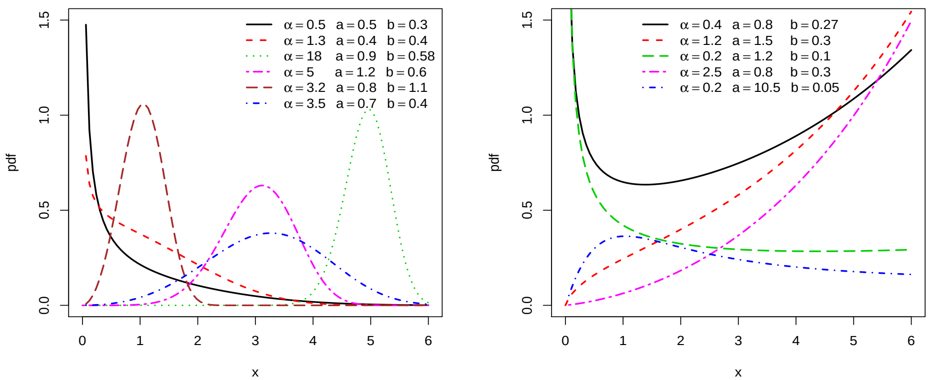

Figure 1: Plots of (a) GKw-E pdfs and (b) GKw-E hrfs for some parametric values with fixed $\lambda = 1$.

Since $Q_G(p) = -(1/\lambda)\log(1-p)$, based on (7), the GKw-E quantile function is given by

$$Q_{GKw-E}(p) = -\frac{1}{\lambda}\log\left[1-\left[1-\left\{1+\gamma_1^{-1}(\alpha,p)\right\}^{-1/b}\right]^{1/a}\right], \qquad p\in(0,1).$$

From this definition, the quartiles and octiles can be determined, as well as skewness and kurtosis, and some results on distributions, as the useful one: for a random variable U following the uniform distribution on $(0,1)$, $Q_{GKw-E}(U)$ follows the GKw-E distribution.

3.2. Linear representation with applications

A result on linear representations of $F_{GKw-E}(x)$ and $f_{GKw-E}(x)$ in terms of exponential functions is presented below.

Proposition 3.1 *We have the following linear representations for the cdf and pdf of the GKw-E distribution:*

$$F_{GKw-E}(x) = \sum_{m=0}^{+\infty} w_m^* e^{-\lambda m x}, \quad f_{GKw-E}(x) = \sum_{m=1}^{+\infty} w_m^{**} e^{-\lambda m x}, \quad x>0,$$

where

$$w_m^* = \sum_{i,j,k=0}^{+\infty} \frac{(-1)^{i+j+k+m}}{\Gamma(\alpha)k!(\alpha+k)}\binom{\alpha+k}{j}\binom{b(j-\alpha-k)}{i}\binom{\alpha i}{m}, \quad w_m^{**} = -\lambda m w_m^*.$$

Proof 3.1 *Let $G(x)=1-e^{-\lambda x}$ and $g(x)=\lambda e^{-\lambda x}$. Then, owing to Proposition 2.3, we have*

$$F_{GKw-E}(x)=\sum_{i=0}^{+\infty} w_i G(x)^{ai}, \quad f_{GKw-E}(x)=\sum_{i=1}^{+\infty} w_i \left[aig(x)G(x)^{ai-1}\right],$$

where

$$w_i=\sum_{j,k=0}^{+\infty}\frac{(-1)^{i+j+k}}{\Gamma(\alpha)k!(\alpha+k)}\binom{\alpha+k}{j}\binom{b(j-\alpha-k)}{i}.$$

Now, for any positive integer i, by virtue of the generalized binomial formula, we have

$$G(x)^{\alpha i}=(1-e^{-\lambda x})^{\alpha i}=\sum_{m=0}^{+\infty}\binom{\alpha i}{m}(-1)^m e^{-\lambda m x}.$$

Therefore

$$F_{GKw-E}(x)=\sum_{i=0}^{+\infty} w_i G(x)^{ai}=\sum_{m=0}^{+\infty} w_m^* e^{-\lambda m x},$$

where $w_m^*=\sum_{i=0}^{+\infty}\binom{\alpha i}{m}(-1)^m w_i$. The desired expansion for the pdf is obtained by differentiating $F_{GKw-E}(x)$. This ends the proof of Proposition 3.1.

Thanks to Proposition 3.1, several structural properties of the GKw-E distribution can be derived. Some of them are described below.

The r-th ordinary moment of the GKw-E distribution is defined by

$$\mu_r'=\sum_{m=1}^{+\infty} w_m^{**}\int_0^{+\infty} x^r e^{-\lambda m x}dx=\frac{1}{\lambda^{r+1}}\Gamma(r+1)\sum_{m=1}^{+\infty} w_m^{**}\frac{1}{m^{r+1}}.$$

Then, we can easily deduce the mean, the variance, the r-th central moment, the coefficient of skewness and the coefficient of kurtosis. The numerical values of these measures for some chosen parameters are collected in Table 2.

Table 2: First four moments, variance skewness and kurtosis of the GKw-E distribution for some parameter values.

(α,a,b,λ)	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CS	CK
$(0.5,0.5,0.5,0.5)$	0.7273	1.6188	5.0408	18.9862	1.0898	2.0021	10.2919
$(2,0.5,0.5,0.5)$	2.8256	10.5036	45.5683	219.8988	2.5192	0.4129	11.6882
$(4,0.5,0.5,0.5)$	4.8084	25.4660	144.9078	872.9899	2.3447	-0.0260	77.1860
$(0.5,2,0.5,0.5)$	2.1594	7.7311	35.2336	186.3619	3.0678	0.9841	5.0394
$(0.5,3,0.5,0.5)$	2.7489	11.1864	56.3355	325.4536	3.6295	0.8140	4.6925
$(0.5,4,0.5,0.5)$	3.2045	14.2449	76.8333	471.4195	3.9758	0.7194	4.8590
$(2,3,0.5,0.5)$	6.0439	40.1030	285.8935	2158.8840	3.5734	0.0468	55.1839
$(4,3,0.5,0.5)$	8.2805	71.2060	632.6335	5784.2410	2.6379	-0.1592	355.3071
$(2,2,1,0.5)$	3.1275	10.8143	40.2792	159.0419	1.0327	-0.0030	136.0614
$(2,2,1.5,0.5)$	2.3533	6.0758	16.8069	49.0889	0.5375	-0.0555	237.7113
$(2,2,1.5,0.1)$	11.7668	151.8967	2100.8630	30680.5800	13.4387	-0.0583	0.1911

It is clear from Table 2 that the GKw-E distribution is numerically versatile in mean and variance. Also, the values of CS reveal that it can be right-skewed, almost symmetrical, and slightly left-skewed. The values of CK indicate that the GKw-E distribution can be mesokurtic, leptokurtic (thin bell shape) and platykurtic (flat bell shape). All these characteristics illustrate a certain flexibility of the GKw-E distribution, which remains attractive for modelling purposes.

In addition, the r-th incomplete moment is obtained as, for $t \geq 0$,

$$I_r(t) = \int_{-\infty}^{t} x^r f_{GKw-E}(x)dx = \sum_{m=1}^{+\infty} w_m^{**} \int_0^t x^r e^{-\lambda m x} dx = \frac{1}{\lambda^{r+1}} \sum_{m=1}^{+\infty} w_m^{**} \frac{1}{m^{r+1}} \gamma(r+1, \lambda m t).$$

The incomplete moments are useful to determine other important mathematical quantities such as the Bonferroni and Lorenz curves, mean residual-life, mean waiting-time, mean deviation about the mean and mean deviation about the median.

4. Estimation and application

In this section, we adopt the GKw-E distribution as a model and consider the estimation of the unknown parameters by the maximum likelihood method. In addition, the convergence of the obtained estimates is investigated through a simulation study and applications are given to two practical data sets.

4.1. Method of estimation

The usefulness of the maximum likelihood estimates (MLEs) in statistical inference is due to their theoretical and practical merits. The log-likelihood function for the vector of parameters $\Omega = \left(a, b, \alpha, \lambda\right)^{\top}$ is given by

$$\ell(\Omega) = n\log(a) + n\log(b) - n\log\left[\Gamma(\alpha)\right] + n\log(\lambda) - \lambda\sum_{i=1}^{n} x_i + (a-1)\sum_{i=1}^{n}\log\left[1 - e^{-\lambda x_i}\right]$$
$$-(b+1)\sum_{i=1}^{n}\log\left[1 - \left(1 - e^{-\lambda x_i}\right)^{a}\right] + (\alpha-1)\sum_{i=1}^{n}\log\left[\left\{1 - \left(1 - e^{-\lambda x_i}\right)^{a}\right\}^{-b} - 1\right] + n$$
$$-\sum_{i=1}^{n}\left\{1 - \left(1 - e^{-\lambda x_i}\right)^{a}\right\}^{-b}.$$

The MLEs of the parameters are defined by $\widehat{\Omega} = \left(\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\lambda}\right)^{\top}$ making maximum the log-likelihood function $\ell(\Omega)$ with respect to Ω. Since they have no closed forms, one can use standard statistical software to approximate them. Also, let us mention that the observed Fisher information for the MLEs can be computed, allowing the construction of confidence intervals for the parameters based on the limiting normal distribution. In particular, this is useful to examine the probability coverage of these intervals through simulation.

4.2. A numerical study

Now, we assess the performance of the maximum likelihood method for estimating the GKw-E parameters by using Monte Carlo simulations. The simulation study is repeated 5000 times each with sample sizes $n = 50$, 100, 200 and the following parameter scenarios are followed: I: $a = 0.5$, $b = 0.5$, $\alpha = 0.5$, and $\lambda = 1$, II: $a = 0.3$, $b = 1.5$, $\alpha = 0.7$, and $\lambda = 2.5$ and III: $a = 1.7$, $b = 0.7$, $\alpha = 0.2$, and $\lambda = 0.3$, IV: $a = 0.1$, $b = 2.5$, $\alpha = 1.1$, and $\lambda = 1.5$,V: $a = 2.5$, $b = 1.7$, $\alpha = 2.5$, and $\lambda = 1$, VI: $a = 1.8$, $b = 1.7$, $\alpha = 2.1$, and $\lambda = 0.1$. Under this setting, Table 3 gives the average biases (Bias) of the MLEs, mean square errors (MSEs) and model-based coverage probabilities (CPs) for the parameters a, b, α and λ. Based on these results, we conclude that the MLEs perform quite well in estimating the parameters. In addition, the CPs of the confidence intervals are quite close to the 95% nominal level. Therefore, the MLEs and their asymptotic results can be adopted to estimate and construct efficiently confidence intervals for the model parameters.

Table 3: Monte Carlo simulation results for the GKw-E distribution: Biases, MSEs and CPs.

		I			II			III		
	n	Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
a	50	−0.015	0.051	0.98	−0.008	0.044	0.94	0.810	14.386	0.85
	100	0.007	0.047	0.97	0.023	0.049	0.95	0.616	4.488	0.90
	200	0.039	0.045	0.96	0.004	0.037	0.95	0.576	2.908	0.95
b	50	−0.140	0.162	0.97	−0.404	1.318	0.90	0.244	3.047	0.97
	100	−0.125	0.127	0.97	−0.217	0.918	0.96	0.307	2.484	0.98
	200	−0.113	0.104	0.95	−0.072	0.477	0.99	0.287	0.977	0.99
α	50	0.153	0.257	0.91	0.465	1.300	0.92	0.452	1.404	0.83
	100	0.084	0.116	0.91	0.307	0.710	0.93	0.225	0.989	0.89
	200	0.046	0.082	0.89	0.306	0.628	0.96	0.139	0.958	0.96
λ	50	1.807	6.527	0.95	2.601	2.726	0.92	0.752	1.324	1.00
	100	1.461	4.742	0.94	1.136	1.129	0.93	0.555	1.002	1.00
	200	1.180	3.136	0.95	0.202	0.847	0.97	0.364	0.743	0.97
		IV			V			VI		
	n	Bias	MSE	CP	Bias	MSE	CP	Bias	MSE	CP
a	50	−0.904	1.154	0.65	0.146	0.535	0.94	0.441	1.253	0.95
	100	−0.665	0.461	0.92	0.164	0.309	0.95	0.194	0.579	0.96
	200	−0.002	0.019	0.97	0.195	0.228	0.97	0.015	0.263	0.99
b	50	−0.032	0.349	0.98	0.172	0.241	1.00	0.018	0.893	0.95
	100	0.014	0.333	0.98	0.053	0.065	0.96	0.072	0.633	0.96
	200	−0.051	0.052	0.96	0.001	0.031	0.97	0.136	0.438	0.98
α	50	0.477	0.480	0.89	0.311	0.163	0.99	−0.158	0.112	0.97
	100	0.270	0.163	0.96	0.271	0.132	0.95	−0.145	0.106	0.96
	200	−0.051	0.052	0.98	0.222	0.100	0.96	−0.148	0.110	0.97
λ	50	0.337	0.601	0.99	−0.062	0.022	0.95	0.179	0.298	0.95
	100	0.214	0.284	0.96	−0.059	0.017	0.96	0.204	0.323	0.96
	200	0.243	0.814	0.98	−0.051	0.011	0.98	0.253	0.392	0.97

4.3. Application

Here, we compare the proposed GKw-E model with well-known models in the fitting of two real data sets.

Application 1. The first data set is reported in Ristic and Balakrishnan (2012). The data represent the annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado from 1900 through 1999. The data are as follows: 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241.

In the statistical literature, several models are appropriate to the analysis of such kinds of data. The most commonly used are the lognormal, generalized logistic (GL), Gumbel, gamma, Weibull and generalized binomial exponential 2 (GBE2) models. Several extensions have also been introduced by this purpose. Here, in order to highlight the potentiality of the GKw-E model, the comparison is made between the GKw-E model and eights notorious models: the Kumaraswamy Weibull (Kw-W) model studied by Cordeiro *et al.* (2010), the Beta Weibull (BW) model elaborated by Lee *et al.* (2007), the exponentiated generalized Weibull (EGW) model by Oguntunde *et al.* (2015), the generalized binomial exponential 2 (GBE2) model introduced by Asgharzadeh *et al.* (2016), the generalized logistic (GL) model, and some classic models, which are the Gumbel, gamma and Weibull models. We estimate the unknown model parameters by the maximum likelihood method (as described in Section 4 for the GKw-E model). For the comparison of the models, we consider three well-known statistics: Akaike information criterion (AIC), Anderson-Darling (A^*), Cramér–von Mises (W^*) and Kolmogorov-Smirnov (K-S) measures, where lower values of these statistics indicate good fits.

Table 4 lists the MLEs and standard errors for the considered models. Table 5 lists the AIC, A^*, W^* and K-S for the considered models. The values of the statistics in Table 5 indicate that the GKw-E model shows small values of the statistics and thus provides the best fit compared to the other models. Figure 2 shows the graphs of the estimated pdf of the GKw-E model over the histogram of the data, along with the graphs of the pdfs of the top four competitors.

Table 4: MLEs and their standard errors (in parentheses) for Precipitation data.

	α	β	a	b	μ	σ	θ	λ
GKw-E	0.2975	-	67.1975	0.1802	-	-	-	0.0261
	(0.1594)	-	(24.7418)	(0.0599)	-	-	-	(0.0072)
Kw-W	0.0228	1.3122	13.4486	0.2461	-	-	-	-
	(0.0053)	(0.2462)	(7.6120)	(0.1229)	-	-	-	-
BW	0.0243	1.4375	12.6298	0.1734	-	-	-	-
	(0.0033)	(0.0193)	(5.5638)	(0.0446)	-	-	-	-
EGW	0.3105	0.7061	0.2357	27.1942	-	-	-	-
	(0.0148)	(0.0117)	(0.0276)	(7.6257)	-	-	-	-
GBE2	9.0774	-	-	-	-	-	0.0222	0.0165
	(1.9764)	-	-	-	-	-	(0.3265)	(0.0029)
GL	13.5845	0.0174	-	-	-8.5348	-	-	-
	(6.8592)	(0.0015)	-	-	(35.1221)	-	-	-
Gumbel	-	-	-	-	139.8754	57.8420	-	-
	-	-	-	-	(6.0596)	(4.7356)	-	-
Gamma	33.2955	5.2761	-	-	-	-	-	-
	(4.7925)	(0.7239)	-	-	-	-	-	-
Weibull	0.0051	2.2608	-	-	-	-	-	-
	(0.0002)	(0.1628)	-	-	-	-	-	-

Table 5: The statistics AIC, A^*, W^* and K-S for Precipitation data.

Distribution	AIC	A^*	W^*	K-S
GKw-E	1137.2320	0.1664	0.0187	0.0421
Kw-W	1138.0280	0.1831	0.0212	0.0430
BW	1137.7220	0.1844	0.0210	0.0429
EGW	1138.7100	0.2045	0.0259	0.0481
GBE2	1138.9210	0.3655	0.0482	0.0573
GL	1143.1390	0.6335	0.0872	0.0565
Gumbel	1139.2900	0.4990	0.0675	0.0640
Gamma	1141.9400	0.7732	0.1088	0.0600
Weibull	1156.2860	1.8272	0.2927	0.0950

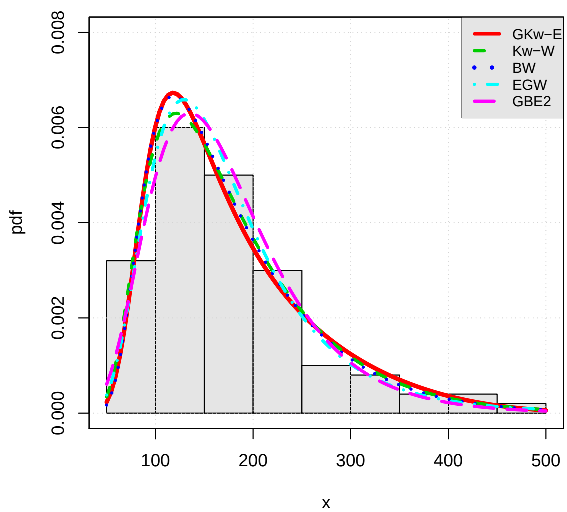

Figure 2: Estimated pdfs of the top models for Precipitation data.

Application 2. The second data set was reported by professor Jim Irish and can be obtained at http://www.statsci.org/data/oz/kiama.html. It is about the Kiama Blowhole eruptions. The data are as follows: 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40,10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

Table 6 lists the MLEs and standard errors for the considered models. Table 7 lists the AIC, A^*, W^* and K-S for the considered models. It is clear that the GKw-E model provides a better fit than the other tested models, because it has the smallest value among AIC, A^*, W^* and K-S. Figure 3 shows the graphs of the estimated pdf of the GKw-E model over the histogram of the data, along with the graphs of the pdfs of the four main competitors.

Table 6: MLEs and their standard errors (in parentheses) for the Kiama Blowhole eruptions data.

	α	β	a	b	μ	σ	θ	λ
GKw-E	0.4154 (0.0545)	- -	17.7076 (0.2513)	0.0481 (0.0072)	- -	- -	- -	0.2063 (0.0046)
Kw-W	0.3410 (0.0026)	0.8685 (0.0022)	10.4397 (0.0083)	0.1396 (0.0168)	- -	- -	- -	- -
BW	0.5484 (0.0025)	0.7937 (0.0025)	13.5819 (4.8229)	0.1336 (0.0177)	- -	- -	- -	- -
EGW	2.5406 (9.4366)	0.3714 (0.2260)	0.7506 (3.0932)	26.1285 (0.8858)	- -	- -	- -	- -
GBE2	1.7325 (0.3190)	- -	- -	- -	- -	- -	0.0048 (0.5680)	0.0350 (0.0111)
GL	21.5045 (6.5526)	0.0473 (0.0048)	- -	- -	-38.5692 (7.8114)	- -	- -	- -
Gumbel	- -	- -	- -	- -	25.6833 (2.8506)	21.8407 (2.3260)	- -	- -
Gamma	24.5722 (4.6509)	1.6207 (0.2623)	- -	- -	- -	- -	- -	- -
Weibull	0.0230 (0.0023)	1.2701 (0.1199)	- -	- -	- -	- -	- -	- -

Table 7: The statistics AIC, A^*, W^* and K-S for the Kiama Blowhole eruptions data.

Distribution	AIC	A^*	W^*	K-S
GKw-E	589.2545	0.4614	0.0530	0.0708
Kw-W	591.0460	0.6231	0.0819	0.0954
BW	591.6412	0.6366	0.0840	0.1023
EGW	595.9134	0.8324	0.1134	0.0946
GBE2	597.3321	0.9009	0.1287	0.1227
GL	612.7799	1.5554	0.2440	0.1517
Gumbel	609.6039	1.5124	0.2361	0.1493
Gamma	595.7988	0.9220	0.1324	0.1215
Weibull	597.8029	1.0058	0.1467	0.1111

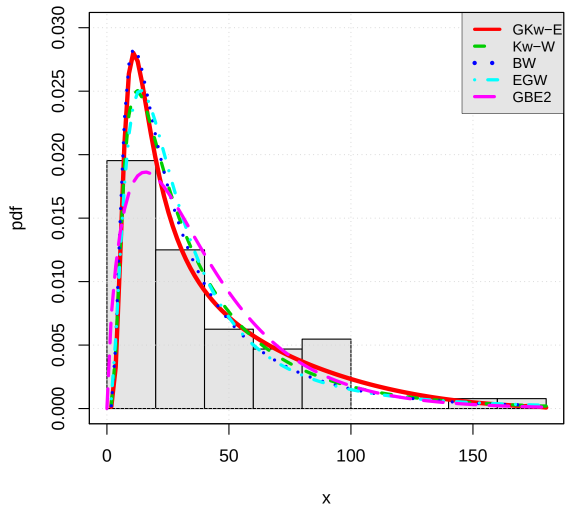

Figure 3: Estimated pdfs of the top models for Kiama Blowhole eruptions data.

5. Concluding remarks

In this paper, we introduce the GKw-G family of distributions, with a focus on a special model, the GKw-E model, defined with the exponential distribution as the parent. A complete theoretical treatment is developed, with a focus on the skewness, kurtosis, analytical comportments of the main functions, moments, stochastic ordering and order statistics. Then, the proposed family is considered from the statistical point of view. The maximum likelihood method is employed for estimating the model parameters. We analyse two practical data sets to demonstrate the usefulness of the new family, with fair comparison to other models. The results are strictly favourable to the GKw-E model. We hope that the proposed family and its generated models will attract wider applications in various areas such as engineering, survival and lifetime data, hydrology and economics.

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Conflict of interest

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